The Forte DEX: An Adjusting Linear Token Bonding Curve

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Abstract

We provide an overview of a novel DeFi market maker called the Adjusting Linear Token Bonding Curve (ALTBC). The ALTBC is a pricing mechanism that automatically updates its price curve in response to arriving trades. Unlike existing AMM's, the ALTBC requires no initial capital and no active management, while automatically increasing collateral through "smart fees" and bounding MEV profitability. Additionally, it is designed to increase local price stability as a function of demand which makes it ideal for projects that wish to promote stable, long-term economies rather than economies fueled by short-term speculation.

In sections 1-3, we illustrate how the ALTBC differs from other market makers. In section 4, we compare the behavior of the ALTBC with existing market makers, using real historical trade data. In the Appendix, we delve more deeply into the same material, giving more mathematical detail and some formal proofs.

1 Related Work and Preliminary Notes

The DeFi liquidity landscape has evolved dramatically over the last decade. Today, there exists a broad range of liquidity protocols, each with different strengths and intended uses. A comprehensive survey of contemporary liquidity protocols is beyond the scope of this whitepaper, but we will consider a few relevant techniques here.[1]

1.1 AMMs and TBCs

Let us begin by reviewing some terminology and common DeFi market making techniques, and discussing how the ALTBC differs from them. In the section below, we discuss the difference between an AMM and a TBC, and show how the ALBTC differs from both.

The most common way to provide decentralized liquidity on chain is through automated market makers (AMMs). Less commonly discussed, but related, mechanisms are token bonding curves (TBCs). The constant product invariant shown in figure 1 underpins many of the most popular AMMs in DeFi. A generalization of constant product market makers (CPMMs) are Geometric Mean Market Makers.[1]

AMMs and TBCs are usually treated separately, with the latter being used to sell tokens that are not yet circulating widely (during a so-called "bootstrapping" phase), or for issuing NFTs one at a time.

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[1] Such a survey is given in [2].

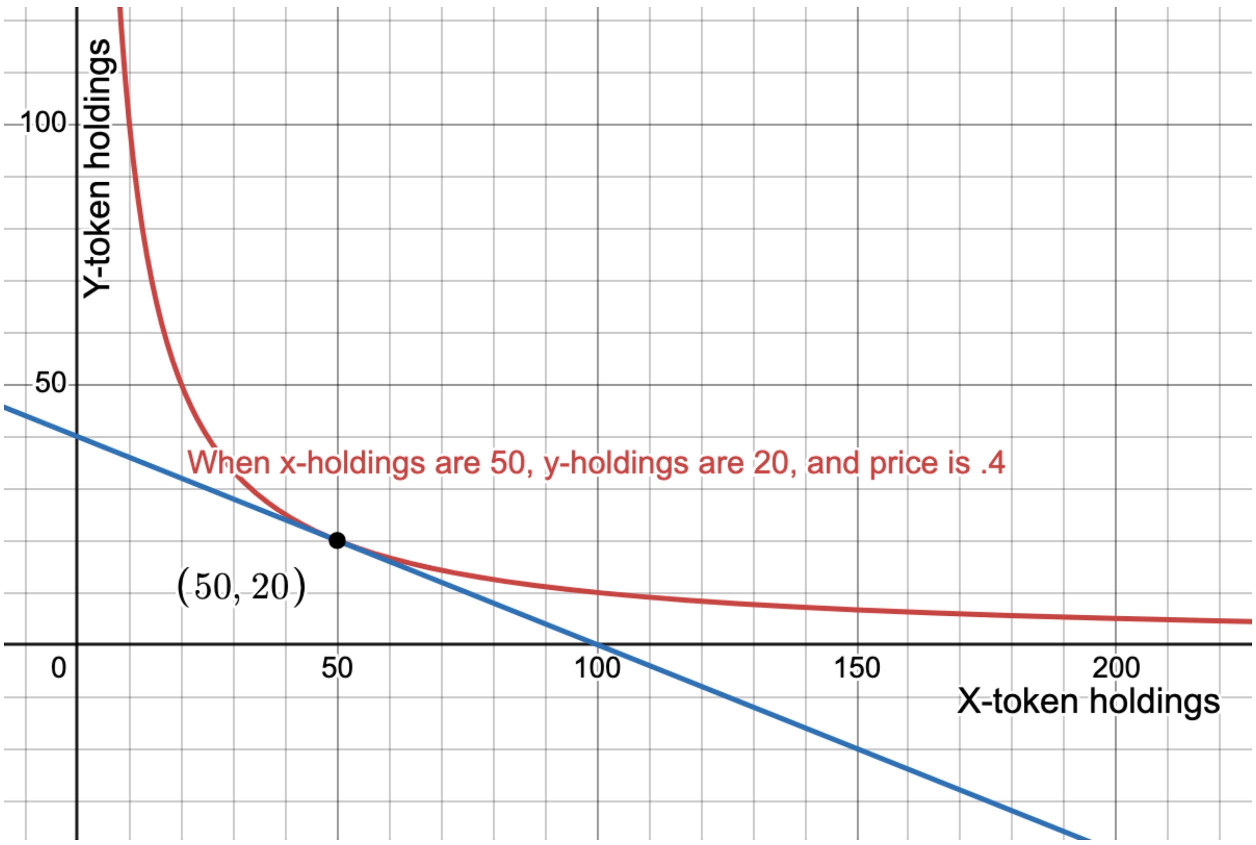


Figure 1: In an AMM, both axes represent token amounts held in the pool. Spot price is given by the negative slope of the tangent line.

However, these two mechanisms are in some cases mathematically equivalent. Specifically, for a finite trading range (for example, a single Uniswap V3 tick range) the pricing rule can be transformed into a single, finite TBC. Therefore, we can think of static AMMs and TBCs as different views of the same basic mathematical structure, although the two are used in different settings and feel psychologically different to the user. See figure 2 for a rendering of a simple TBC.

1.2 Updating Price Rules

We can say, therefore, that the two most common liquidity providing techniques – AMMs and TBCs – are mathematically equivalent. Let us now distinguish between between these – both of which rely on invariant bonding curves – and *updating* price rules like the ALBTC, whose logic can change over time.

The ALBTC is composed of a line whose parameters constantly adapt in response to the arrivals of trades. With every buy, for example, the slope of the line (corresponding to the sensitivity of the price to trade size) goes down. In this way, price becomes less volatile with increased demand. In this paper, we depict the price function in the usual TBC-space (x-axis is amount minted, y-axis is price), but it must be remembered that any such picture shows a snapshot in time, since the true price function is constantly changing.[2]

The ALBTC is not the first updating price rule. Several projects, for example, have used price oracles to update their pricing curves, often with the goal of mitigating losses to arbitrageurs. Arrakis's HOT AMM, the Lifinity protocol, swaap.finance's Matrix-MM, and DODO's "proactive market maker" all rely on price oracles to keep quotes from going stale and reduce a pool's vulnerability to informed traders.

These projects, and others like them, are updating price rules. That is to say, their price rules can vary over time (unlike in Uniswap V2 or a standard TBC). However, it is important to distinguish between their approach, which relies on oracles and where updates can happen independently of trades, and the one in question here. Unlike these oracle-based approaches, the ALTBC updates exclusively in response to trader activity, not external price feeds. Moreover, its updates optimize for economic health and capital accumulation, rather than arbitrage mitigation. This makes the mechanism fully decentralized and community-oriented. Curve Finance has released two projects that are not oracle-based, and whose curve updates, like ours, occur

[2]We term such snapshots the "intermediate TBC."

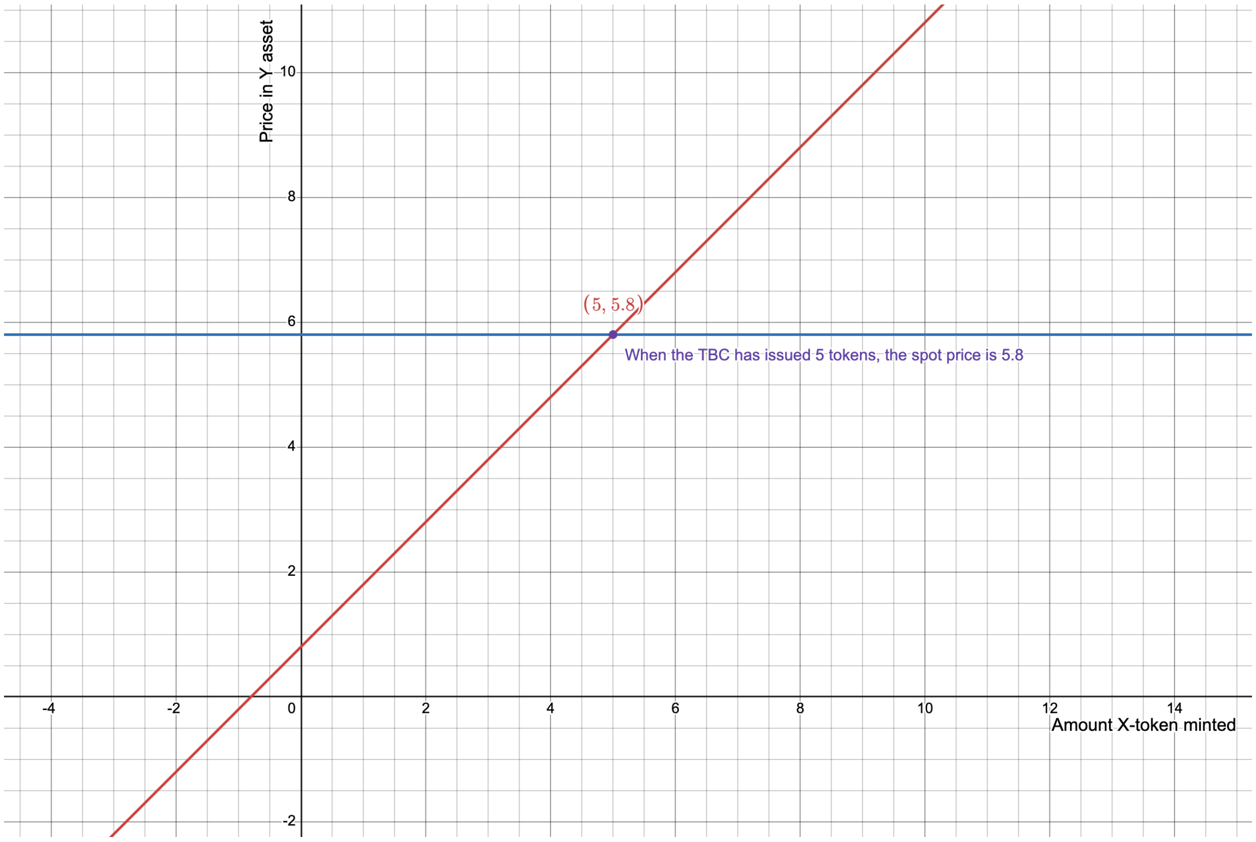


Figure 2: A TBC maps amount minted (x-axis) to price (y-axis).

in response to arriving trades.[3] These projects achieve a high degree of control over price impact; although they are updating price curves that, like the ALBTC, react to arriving trades rather than oracles, they solve very different problems and have different use cases.

1.3 Use Cases for the ALBTC

This mechanism can be used to bootstrap liquidity without providing initial collateral as is required with a conventional AMM. For this reason, it is ideal for new projects where tokens have not had time to circulate widely.

It also has a unique way of accruing capital without relying on separate buy and sell functions, and even before standard fees are imposed.[4] This makes the ALBTC a good fit for community-oriented tokens where the market maker wishes to incentivize behaviors that enrich the community as a whole.

Finally, the ALBTC will produce less and less volatility as more x-token circulates. This makes it a good home for projects that wish to promote thriving, stable economies rather than economies fueled by short-term speculation.

1.4 Operating in a closed system

Let us further note that, in the discussion to follow, it should be understood that the ALBTC is the only market for x-token. Thus all considerations that depend upon external markets are not relevant here.[5]

[3] Respectively, StableSwap (for trading stablecoins) and CryptoSwap (a more complex mechanism for all purpose cryptocurrency trading). See [3] and [4].

[4] We term these capital accrual properties “smart fees.”

[5] Loss-versus-holding, for example, depends on having another place to liquidate portfolio holdings. If we consider the market owner the sole liquidity provider, this metric cannot apply.

2 ALTBC Design Overview

2.1 General Approach

The ALTBC is composed of a line whose slope and intercept adapt to the arrival of trades. Thus, the basic price function is a simple line:

$$f_n(x) = b_n x + c_n$$

Why is the price function f subscripted? Because for any given moment in the curve's lifetime, a different line will apply, depending on the history of updates. The logic governing these updates is detailed in the Appendix, but note that these adjustments ensure three key things:

1. **price** increases as more tokens enter circulation,
2. **price impact** decreases as more tokens enter circulation (inverse volatility), and
3. **collateral**[6] **continually accrues** with each trade, even when supply returns to previous levels. See Section 3.3.

2.2 ALTBC Constant Variables

In this section, we discuss the quantities that the ALTBC uses. Some of these are constants set at instantiation, whereas some are updated. Here, we discuss these variables one at a time and provide some helpful discussion.

We begin with variables that are set once upon instantiation and never altered again.

Description	Variable	Definition or Bounds
Initial tokens deposited into TBC	x_{add}	$x_{\text{add}} > 0$
y-intercept upon TBC initialization	p_{lower}	$p_{\text{lower}} \geq 0$
Vector Field Parameter	V	$V > 0$
Initial Concentration Parameter	C_0	$C_0 > 0$
Initial minimum value of x	$x_{\text{min}}^{(0)}$	$x_{\text{min}}^{(0)} > 0$
Liquidity units assigned to deployer at pool initialization	W_0	$W_0 > 0$
Deployer's inactive liquidity units at launch	W_0^I	$W_0 > W_0^I > 0$
Initial trading fee	ϕ_0	$\phi_0 \geq 0$
Initial protocol fee	ψ_0	$\psi_0 \geq 0$

2.3 ALTBC Maintained Variables

We now introduce the quantities that the ALTBC maintains over time, updating with each transaction. See table **??**.

2.4 Discussion of key variables

Now, let's gain some intuition about how the mechanism works by discussing some of the variables.

x values

The x_{add} value represents the maximum number of x-tokens the ALTBC can sell. It is also, equivalently, the amount of x-token deposited into the pool upon instantiation. x_{min} is the lowest possible x-value the price curve can achieve. The sum of these two numbers is the upper bound on x for the ALTBC. Transactions that would push the x-value outside this realizable trading region $(x_{\text{min}}, x_{\text{min}} + x_{\text{add}})$ fail.

Concentration Parameter C

The concentration parameter C is a number that informs the slope update's sensitivity to order size, that it is set at initialization and changes only with liquidity changes (see below at Section 2.11), and that it is initially determined by the parameter setting algorithm discussed below.

[6] Here, we define collateral as the value in y-token of all the x-token that traders have purchased from the market.

Description	Variable	Initial value
Vector Field Param.	V	V (constant)
Value of x before n–th txn	x_n	$x_0 = x_{\text{min}}^{(0)}$
Area under TBC curve before n–th txn	D_n	$D_0 = \mathbf{D}(x_{\text{min}}^{(0)}, \mathbf{b}(0, C_0, V_0), p_{\text{lower}})$
Parameter b before n–th txn	b_n	$b_0 = \mathbf{b}(x_{\text{min}}^{(0)}, C_0, V_0)$
Parameter c before n–th txn	c_n	$c_0 = \mathbf{c}(x_{\text{min}}^{(0)}, \mathbf{b}(x_{\text{min}}^{(0)}, C_0, V_0), \mathbf{b}(0, C_0, V_0), p_{\text{lower}})$
Spot price before n–th txn	p_n	$p_0 = \mathbf{p}(x_{\text{min}}^{(0)}, b_0, c_0)$
Concentration param.	C_n	C_0
Minimum value of x	$x_{\text{min}}^{(n)}$	$x_{\text{min}}^{(0)}$
Maximum value of x	$x_{\text{max}}^{(n)}$	$x_{\text{max}}^{(0)} = x_{\text{min}}^{(0)} + x_{\text{add}}$
Sum of NFT amounts	W_n	W_0
Deployer's inactive liquidity	W_n^I	W_0^I
LP fee balance adj.	Z_n	$Z_0 = 0$
Trading fee ratio	ϕ	ϕ_0 (constant)
Protocol fee ratio	ψ	ψ_0 (constant)
Claimable trading fees per unit active liquidity	Φ_n	0
Total protocol fees per unit active liquidity	Ψ_n	0

Table 1: Summary of TBC Maintained values and Initial Values

Vector Field Parameter V

The parameter V controls the sensitivity of the slope field to trade size. A higher V value will, in general, produce more price volatility. This can be seen by examining the equation for the slope field (the equation that maps x-value to the slope of the pricing line):

$$s(x) = \frac{V}{x + C}$$

Holding x and C constant, we can easily see that slope is increasing in V. Thus, V is a tuning parameter for the ALTBC; higher values will lead to more price fluctuation. Like C, V is an output of the parameter setting algorithm.

2.5 Functions

Below are functions that we will use throughout:

Function	Equation	Meaning
$\mathbf{D}(x_{n+1}, b_n, c_n)$	$\frac{1}{2} b_n x_{n+1}^2 + c_n x_{n+1}$	Area under curve.
$\mathbf{b}(x_n, C_n, V)$	$\frac{V}{x_n + C_n}$	Slope of pricing function.
$\mathbf{c}(x_{n+1}, b_{n+1}, b_n, c_n)$	$\frac{1}{2}(b_n - b_{n+1}) x_{n+1} + c_n$	Floor price.
$\mathbf{p}(x_n, b_n, c_n)$	$b_n x_n + c_n$	Spot price.
$\mathbf{L}(x_n, b_n, c_n, x_{\text{min}}^{(n)}, C_n, V)$	$\frac{1}{2} x_{\text{min}}^{(n)} \left(b_n x_n + 2c_n + V \cdot \ln \left(\frac{x_{\text{min}}^{(n)} + C_n}{x_n + C_n} \right) \right)$	Limiting TBC Primitive.
$\mathbf{h}(x_n, b_n, c_n, x_{\text{min}}^{(n)}, C_n, V, W_n, W_n^I, \Phi_n, Z_n)$	$\frac{\mathbf{L}(x_n, b_n, c_n, x_{\text{min}}^{(n)}, C_n, V) + Z_n}{W_n - W_n^I} + \Phi_n$	Revenue Parameter.

Table 2: ALTBC functions

2.6 Cost Functions

In order to actually execute a transaction, we need more than just the spot price: we need functions that take in a transaction amount, and output the correct cost for that order size and trade direction. To evaluate the cost of a buy or sell order of x-token, we must:

1. Calculate the value (in y-token) of current outstanding supply.
2. Calculate the value (in y-token) of (current outstanding supply + transaction amount).
3. Take the difference between the two. This gives us the cost (in y-token) of transacting the proposed amount.

To evaluate the cost of a buy or sell order of y-token, we must:

1. Calculate the value (in x-token) of current collateral.
2. Calculate the value (in x-token) of (current collateral + transaction amount).
3. Take the difference between the two. This gives us the cost (in x-token) of transacting the proposed amount.

These functions are listed in Table 3.

Variable	**Description**	**Input**	**Output**
F_{n}	Cost Function	x-token transaction amount	value in y-token
F_{n}^{-1}	Inverse Cost Function	y-token transaction amount	value in x-token

Table 3: Cost Functions

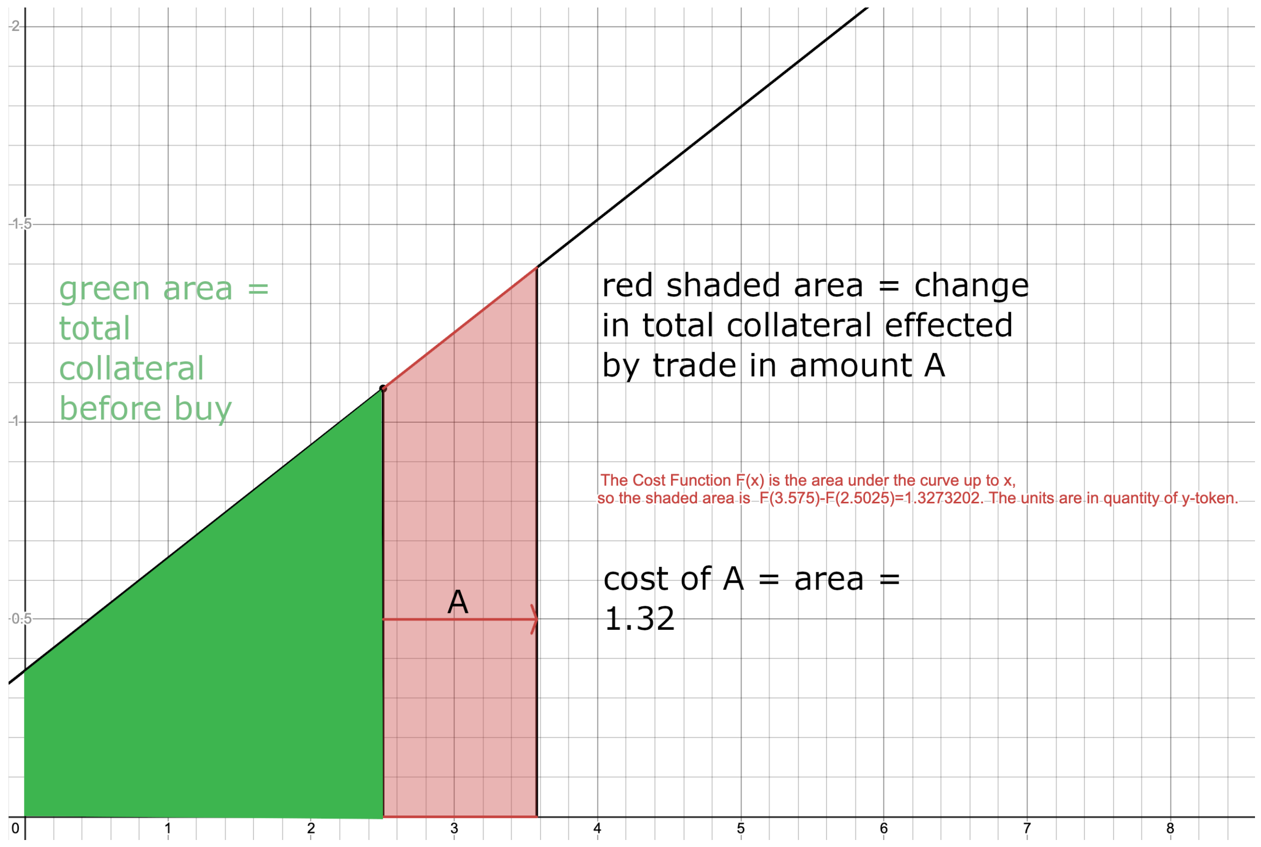


Figure 3: The cost function used to evaluate the cost of a buy order.

2.7 Level Curves, the Concentration Parameter C, and Generalizing the TBC

Let us now build some intuition for how the line adapts. In the ALTBC, every x value is mapped to a unique slope value. This is achieved with a slope field, which is defined as the following family of level curves:

$$g(x) = \ln(x + C) + K$$

K simply shifts the log curves up or down. It appears here to help illustrate the family of level curves, but is not a genuine ALTBC variable.

C parameter

We term C the "concentration parameter" because it controls the degree of concentration of price in a given supply range. C achieves this by shifting the level curves left and right, effectively controlling the slope update's sensitivity to changes in outstanding supply. Geometrically, higher C values shift the log curves to the left. This means that the level curves are closer to flat at lower values of supply, which produces greater price stability or, equivalently, less sensitivity in slope updates. See figure 4.

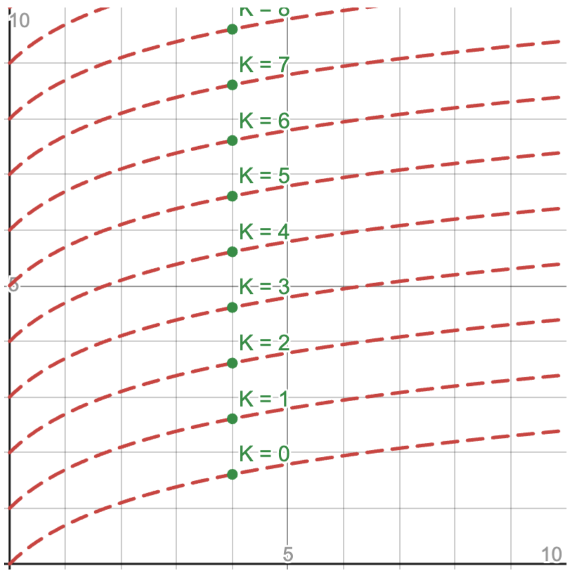
Description	Variable	Bounds
Concentration Parameter	C	$C > 0$

Table 4: C: The Concentration Parameter

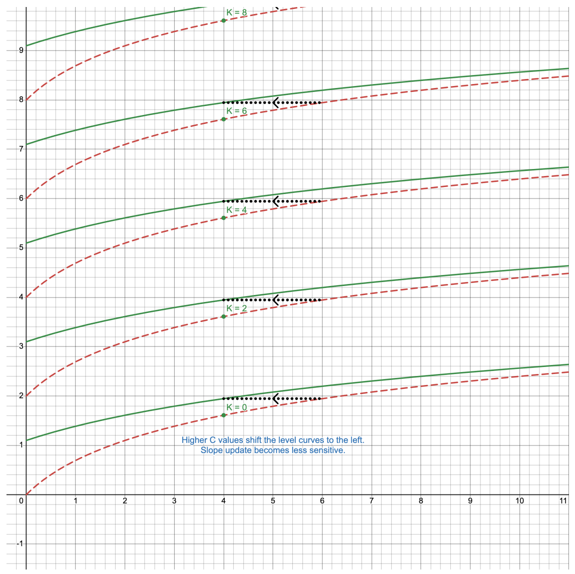
Slope is a deterministic function of supply, generalizing the TBC

The line $f_n(x)$ will be tangent to the level curve which passes through the point (x_n, p_n),[7] where x_n is the current x-value and p_n is the current spot price. Note that this means that while a given x-value does not pick out a unique price (because there are many lines with the correct slope), it *does* pick out a unique slope

[7]The point of tangency will be the point (x_n, p_n).



(a) Level curves with C = 1



(b) Level curves shifted (C = 3). Shifting the log curves to the left means that they are closer to level at lower x-values.

Figure 4: Comparison of level curves for different values of C

b_n. We term the property of a unique slope for every supply value the "path independence of slope" (this is an important property of the ALTBC; see section 3).

Note that the path independence of slope property means that slope, not price, is a deterministic function of supply. Another way to say this is that **price impact** rather than **price** is a function of supply. In this respect, we consider the ALTBC to be a generalization of the idea of a TBC.

2.8 Limiting TBCs vs Intermediate TBCs

Let us introduce two concepts:

- An **intermediate TBC**, which is simply a way of referring to an individual $f_{\mathrm{n}}(x)$ pricing function.
- A **limiting TBC**, which is a curve representing optimal obtainable prices for the trader. These prices are obtainable if we grant the trader the ability to split their orders into infinitely many smaller sub-orders; it is not, in other words, generally going to resemble the actual price curve yielded by the ALTBC.[8] Nevertheless, the concept gets to the heart of the ALTBC and is important to understand. See section 3.5.

The limiting TBC at any point is given by the equation

$$f_{\mathrm{lim}}(x) = p_n + \frac{1}{2}\left(\frac{x}{x+C} - \frac{x_n}{x_n+C}\right) + \frac{1}{2}\ln\left(\frac{x+C}{x_n+C}\right)$$

where x_n is the x value at time n, p_n is the current price, and C is the concentration constant. An example of an intermediate TBC, and its accompanying limiting TBC, is shown in figure 5.

2.9 Liquidity Positions

Ownership of the pool (by developers and liquidity providers) will be represented by NFTs. These NFTs will store the following information:

$$\mathbf{NFT}(\{\texttt{amount}\colon a,\ \texttt{last\_revenue\_claim}\colon r\}). \tag{1}$$

The amount a stores the total number of units of liquidity owned by the NFT. The `last_revenue_claim` quantity stores a number that benchmarks the total amount of liquidity at the time of the liquidity add in

[8]Note that in any implementation, the minimum trade size will be a single "micro unit" of the token in question (e.g. $1/10^{18}$ ETH). Thus the Limiting TBC is not achievable in the discrete setting of the EVM.

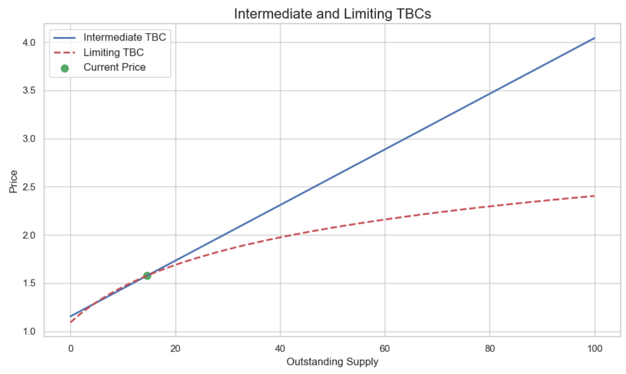


Figure 5: An example intermediate TBC and limiting TBC. The limiting TBC in red shows the prices a trader would obtain if infinitesimal order splitting were possible.

question, ensuring that the LP, when they collect revenue or withdraw liquidity, only redeems fees accrued since the moment of the liquidity add. The value of the `last_revenue_claim` field of the NFT j at step n is denoted by $r_j(n)$. Note that these values are not tracked by the pool since they are stored in the corresponding NFT itself.

2.10 Initializing the TBC

When the pool is initialized, the state variables are defined as described table 1. In addition, an amount x_{add} of tokens must be deposited into the pool. For reference in what follows,

$$D_0 = \mathbf{D}(x_{\text{min}}^{(0)}, \mathbf{b}(0, C_0, V_0), p_{\text{lower}}) \ . \tag{2}$$

Two NFTs must be minted and given to the pool deployer upon initialization of the pool.

The first will have its `amount` field value equal to $W_0 - W_0^I$ and its `last_revenue_claim` field value equal to D_0. This NFT is represented as:

$$\mathbf{NFT}(\{\texttt{amount}\colon W_0 - W_0^I,\ \texttt{last\_revenue\_claim}\colon D_0\}). \tag{3}$$

The second, which will henceforth be referred to as the **inactive-fee NFT** (and is unique in this way), will have its `amount` field value equal to W_0^I and its `last_revenue_claim` field value equal to ∞. This NFT is represented as:

$$\mathbf{NFT}(\{\texttt{amount}\colon W_0^I,\ \texttt{last\_revenue\_claim}\colon \infty\}). \tag{4}$$

Note that ∞ can be replaced by any appropriate number or object that correctly handles the following three actions for the inactive-fee NFT:

1. Extracting revenue: the transaction should be immediately reverted (described below).
2. Withdrawing liquidity: the transaction is allowed with some modifications (described below - for example, step 8 in Liquidity withdrawals).
3. Adding liquidity: the transaction should be immediately reverted (described below).

2.11 Liquidity Adds and Withdrawals

It may seem counterintuitive to speak of adding liquidity to a TBC. Nevertheless, as previously observed, a TBC is mathematically equivalent to an AMM defined for a finite number of tokens. Thus it is possible to add liquidity to a TBC, and it is therefore possible to deploy the ALTBC as a general purpose pricing mechanism.

In order to add liquidity to the ALTBC, it is necessary to consider the ratio of assets being proposed by the prospective liquidity provider. A given ALTBC state implies a ratio of one asset to the other. So, for example, if the price of one x-token is 10 y-token, a liquidity add in the amounts of 10 y-token and 1 x-token would be exactly proportional.

The ALTBC accepts only proportional liquidity adds. When a liquidity provider offers tokens in quantities out of proportion, the pool will determine proportional amounts, and return the difference to the LP in question.

Tracking Liquidity Positions and Pool Ownership

The introduction of liquidity positions requires a mechanism for tracking pool ownership. As mentioned above, this is achieved by issuing NFTs that track the value of a given liquidity position and the cumulative amount of value so far redeemed by the NFT owner in question.

2.11.1 Example Liquidity Add Flow

Liquidity Add Flow	
Step	**Description**
1	Liquidity Provider submits a call to the ALTBC to add liquidity amounts in quantities A and B.
2	The ALTBC adjusts A and B to be in the correct proportion (A*, B*), returning the unused portions of the submitted amounts to the LP.
3	A* and B* are added to the pool.
4	The ALTBC state changes to reflect these additions.
5	An NFT is minted to the Liquidity Provider (or updated, if an existing NFT is provided) indicating the value of the created position.

Table 5: Liquidity Add Flow

2.12 Pool Ownership and Revenue Extraction

Note that the existence of these LP tokens implies that ownership of the pool is shared among liquidity providers (the amount of value stored in the NFT represents the portion of the entire pool that belongs to the NFT owner). Holders of these NFTs, therefore, can use them in two different scenarios:

1. When pulling liquidity from the pool.
2. When distributing accumulated excess capital from the pool to liquidity providers.

The liquidity removal is straightforward; the LP can pull out the amount of liquidity originally added, along with fees accrued since the time the liquidity position was opened (using the **last_revenue_claim** quantity).[9]

Extracting Excess Capital

The second use for the LP NFT – regarding excess capital – is less straightforward. What is this excess capital? Recall the remark above, that for a given x-token value, collateral continually accrues to the market owner(s) with each transaction.[10] Most of that collateral will supply liquidity for trades in the reachable domain of the ALTBC – that is, $x_{\min} \le x \le x_{\max}$. But some of that increased collateral will **not** be accessible to traders. That is, even if all available x-token were sold back to traders, some collateral would be left over.

This is illustrated in figure 6. In that illustration, we see an initial price function $f_{\mathrm{n}}(x)$. Suppose we begin at $x = x_{\min}$, some trading takes place, and then there is a large selloff so that we return to $x = x_{\min}$. Our new price curve will be higher: the ALTBC will look like the green green line $f_i(x)$. Much of the accumulated collateral will have been issued back to traders during the selloff, but an amount will remain even after the last token has been sold back to the market. This amount is highlighted in green, and it will become the property of the market owner(s).

It is this quantity that must be shared by market owners (i.e., liquidity providers), in amounts corresponding to the their NFTs. Thus, the LP NFTs must also track the cumulative amount of revenue so far paid out to LP position holders, so that they can redeem up to (but not more than) the amounts they own. Table 6 shows the basic information that an LP NFT must store.

[9] Note that this withdrawal function will always pay the LP with asset amounts in the proportion implied by current AMM state.

[10] We term this property the monotonicity of spot price for given supply value; see Appendix.

Field	Meaning
`token_id`	Unique identifier for the LP NFT.
`amount`	Represents the portion of liquidity ownership associated with this NFT.
`last_revenue_claim`	Ensures the correct amount of excess capital is paid out to NFT holder.

Table 6: LP NFT Fields

Withdrawing Liquidity

When a liquidity provider wishes to withdraw a portion of the liquidity stored in an LP NFT, three things must happen (see Appendix for a more detailed explanation):

1. An amount of x-token and y-token corresponding to the value of the NFT is issued to the LP.
2. An amount of excess capital belonging to the LP is issued to the LP.[11].
3. The ALTBC state is updated to reflect the resulting change.

As mentioned above, LPs can use their LP NFTs to redeem value in two ways: they can withdraw the liquidity they originally added, or they can redeem the amount of excess collateral to which their LP position entitles them.

These two functions are separate, but related. An LP can request a revenue payout without withdrawing any liquidity, but a liquidity pull automatically triggers a proportional revenue payout.[12]

In either case, whenever revenue is paid out, the LP NFT in question must be updated (in the **last_revenue_claim** field) so that the LP can never redeem more revenue than is owed.

2.13 Note on Interpreting the Pricing Curve

It may be useful to think about how to interpret the variables stored in the ALTBC.

Let us begin with x_n. Typically, in TBC space, we think of the x-axis as representing the total number of tokens minted by the TBC. That is the case for the ALTBC – until someone chooses to add liquidity to the pool.

In a TBC that accepts external liquidity changes, x_n does **not** represent total circulating supply. Instead, the proper way to think of x_n is the following:

> In a TBC that accepts liquidity additions and withdrawals, x_n is the **amount of x-token required to drain the pool of all its collateral.**

Now, let us consider the user-supplied variable x_{add}. This represents the initial maximum number of tokens for sale by the ALTBC. However, it is **not** the highest value that x_n can achieve. Remember that the reachable trading region is offset by the presence of x_{min}. Thus the highest reachable x-value is actually x_{max}, which is the sum of x_{min} and x_{add}.

Finally, let us reiterate that the value of x_{max} is subject to change with the addition of new liquidity. We can interpret this as meaning that, when an LP adds liquidity to the pool, more x-token is now for sale (and, correspondingly, more y-token is held as collateral.)

2.14 Setting initial ALTBC parameters

Above, we have discussed the various parameters used to initialize the ALTBC, and which parameters it maintains over time. Using those variables, however, it is not immediately obvious how to set initial values. In order to facilitate the selection of reasonable initial values, we have developed an algoritm that requires the user to specify four easily interpretable values:

1. `l`: the specified multiple of initial price amount.
2. `Q`: the cost to the exploiter[13] of pushing the initial price up by a factor of l.

[11] This amount must take into account the cumulative amount of revenue so far paid out to the LP in question

[12] The reason for this is subtle: if the liquidity were paid out without the corresponding revenue payout, it would become impossible to recover the revenue in question.

[13] See section 3.6

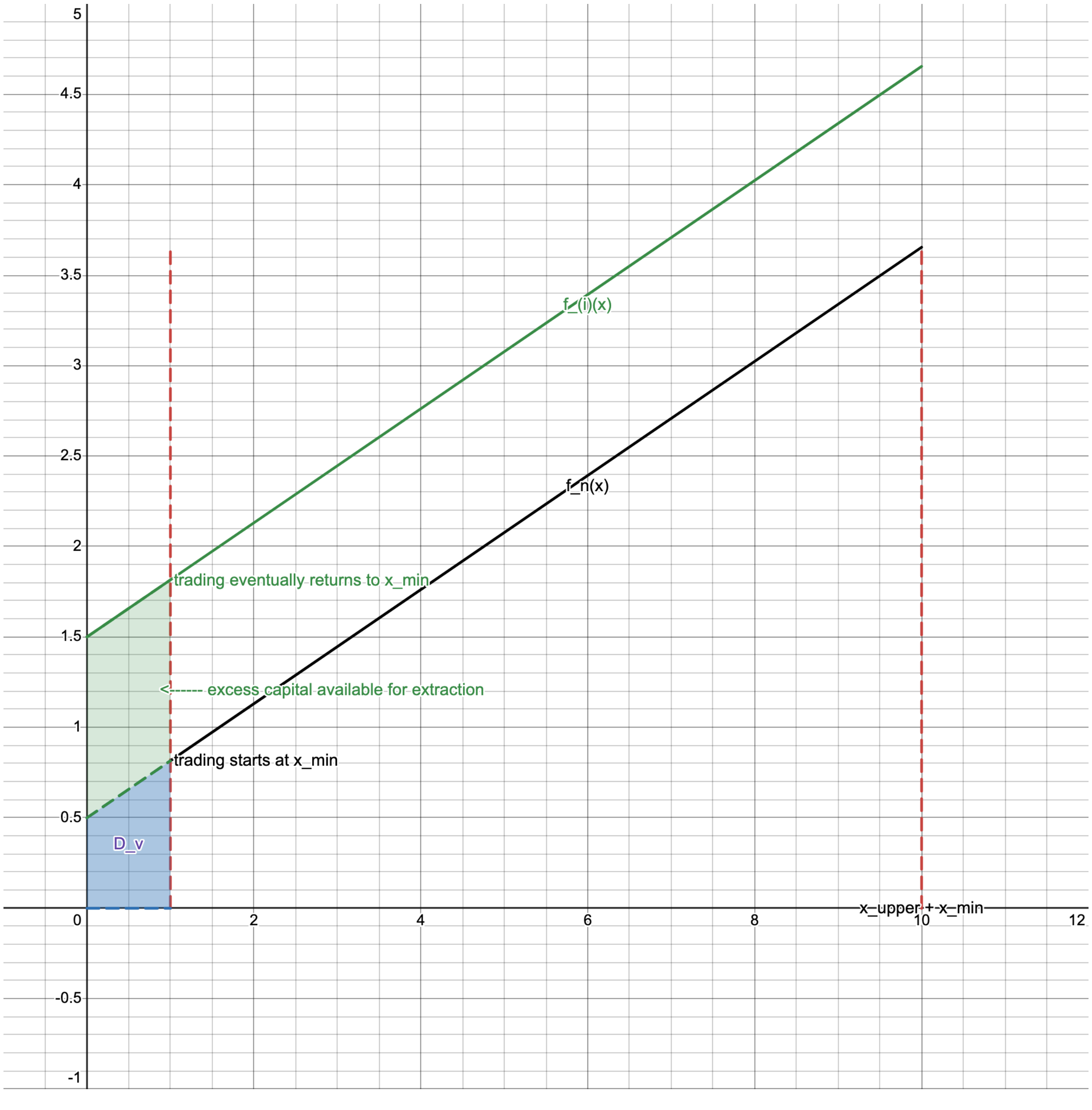


Figure 6: Excess capital accrues at x-values below $x_{\min}$. This excess value is available to market owners via the revenue function.

3. `T_target`: a certain number of tokens to sell, such that item (4) is true.
4. `p_target`: the minimum price of the asset after `T_target` tokens have been purchased.

In practice, a user submits these four numbers, and the algorithm determines all the appropriate internal variables for the initialization of the ALTBC. The reasoning behind these variables, and exactly how they are used to compute the internal parameters of the ALTBC, is explained in the Appendix.

3 Notable Properties

In this section, we focus on several of the advantages and notable properties of the ALTBC.

3.1 Directional consistency of spot price.

The price rule changes as traders buy and sell over time. Specifically, the slope of the TBC line decreases as circulating supply increases. In spite of this decrease, the ALTBC will always obey the law of supply and demand; namely, the price will always increase with buy trades, and will always decrease with a sell trades.

3.2 As x increases, volatility decreases.

As x values go up, the slope of the line will decrease. This means that the price impact of a given trade will be smaller and smaller the more x-token is circulated. Note that this contour (volatility decreasing as more x-token is in supply) is the opposite of a Uniswap pool. See section 4.

3.3 Price and collateral are monotonic at any given x-value.

Suppose the TBC has minted 100 tokens at transaction #10. The current price f(x) = 200. Then, 3 transactions arrive: a buy of 10, a sale of 15, and a buy of 5. The x-value traces the path 100, 110, 95, 100 – a round trip returning to 100. What has happened to the collateral and the price? This property says that both are always higher at transaction #13 than at transaction #10.

3.4 Innate MEV sandwich attack mitigation

A sandwich attack operates by "sandwiching" a given trade by two trades designed by the exploiter. The first trade pushes the price up. The "sandwiched" trade settles at a less favorable price than anticipated. Then, the attacker sells at a slightly better price than they purchased at, pocketing the difference at the sandwiched trader's expense.

No AMM can completely remove the profitability of these kinds of attack.[14] The ALTBC, however, establishes an upper bound on their profitability. The ALTBC's stability, along with its collateral preservation property, make it so that only a finite range of sandwich attack sizes are profitable for a given target trade. The section below illustrates this property.

3.4.1 Illustration of innate MEV sandwich attack mitigation.

Let us fix the target trade at 1 x-token coins. Let us also fix the various parameters internal to the ALTBC and designate an initial x-value for the ALTBC of 2.

The attacker now has just one decision to make: the magnitude of the sandwich attack itself. Let us designate this magnitude as S. The attacker wishes to extract as much profit as possible, i.e. the difference between the initial purchase price for S tokens and the subsequent sale revenue for S tokens should be maximized (with the sale price being greater). In the ALTBC, some S values will yield profits (see figure 8). However, if the sandwich size is too large, the sandwich attack will produce a loss (see figure 9).

This offers a crucial advantage over conventional liquidity solutions. Note that in the case of Uniswap V2, MEV is mathematically unbounded.[15] In the ALTBC, however, there is an upper limit on how much a sandwich attacker can earn, even before fees and slippage tolerances enter the picture. Figure 7 shows this bound clearly by plotting the attacker's profit and loss as a function of sandwich attack size. See Appendix A.5 for the derivation of MEV as a function of attack size.

[14] Except one that provides a constant price. See [5]

[15] In practice, user-supplied slippage tolerances are required to prevent this fact from causing problems.

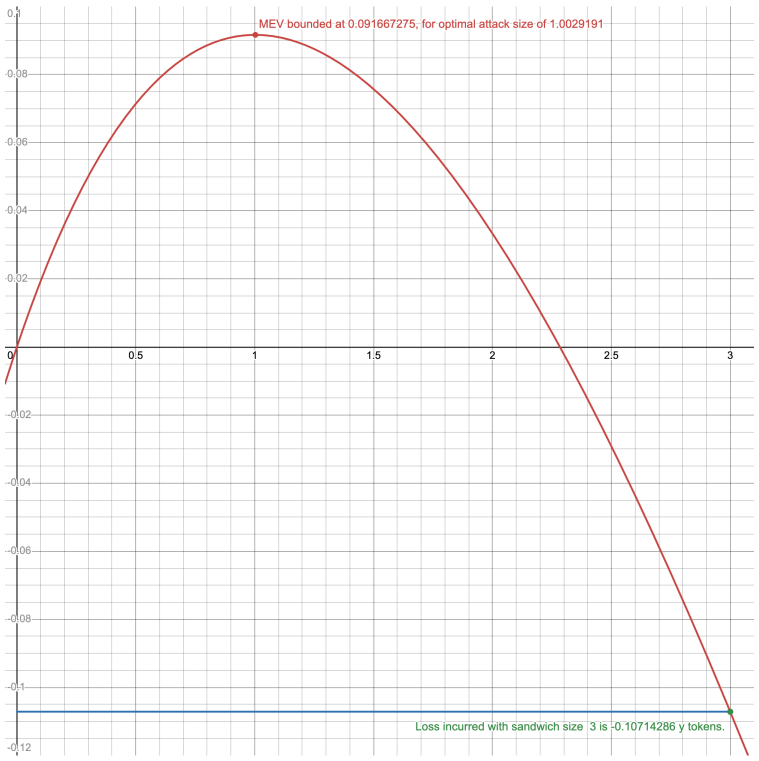


Figure 7: MEV as a function of sandwich attack size, fixing the victim trade size and other parameters. An attack size 1.003 is maximally profitable, whereas an attack size 3 yields a loss.

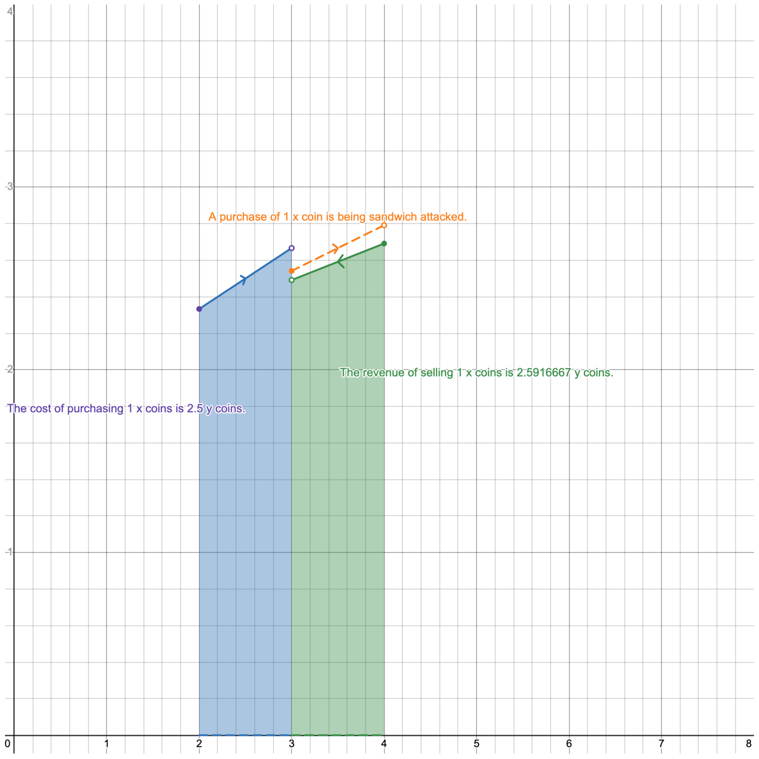


Figure 8: Fixing victim trade size, starting supply, and other parameters, a sandwich attack of size 1 is profitable.

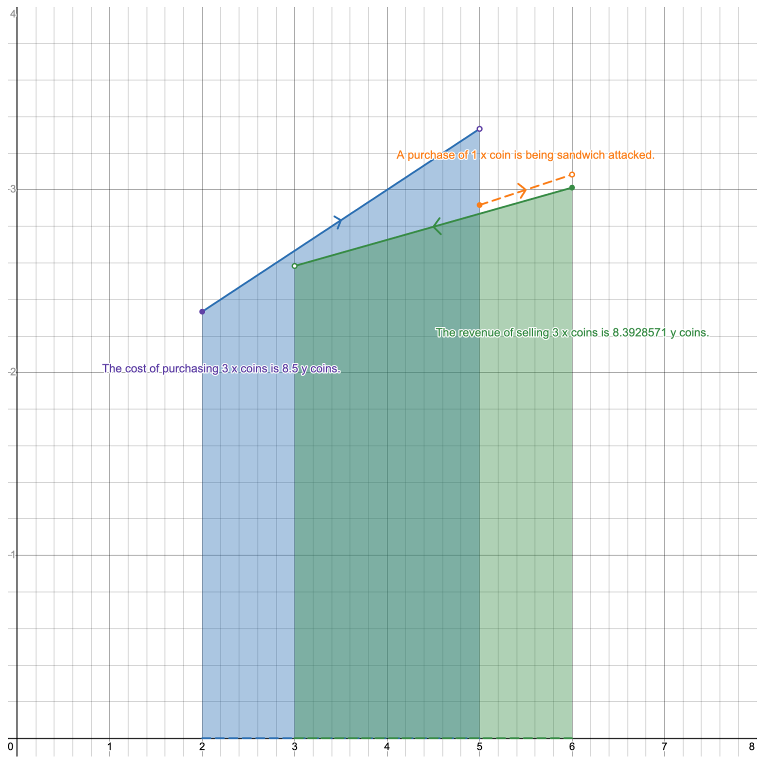


Figure 9: Fixing victim trade size, starting supply, and other parameters, a sandwich attack of size 3 produces a loss for the attacker.

3.5 Bounded incentive for trade-splitting

Recall that the main feature of the ALTBC is that the slope of the line decreases as circulating supply increases. A natural, but not immediately obvious, consequence of this property is that for any given trade, it will be in the trader's interest to split that order up into many smaller orders. We can think of this feature as the ALTBC making a compromise. It sacrifices a small amount of "size consistency"[5], in order to make significant gains in flexibility, price stability, security against MEV attacks, and profitability. All liquidity solutions make tradeoffs with these properties. In the section below, we explain how the ALTBC engenders this incentive, and why we think it does not represent a meaningful downside to the design.

3.5.1 Illustration of Bounded Incentive to Split

Consider an order of some fixed size, split up into n separate orders. As n grows, the number of times the slope of the price curve decreases also grows. Every price update represents an improvement, from the perspective of the trader, over a smaller number of order partitions. Thus it is optimal for the trader to split orders into infinitely many sub-orders.[16] This infinite splitting yields an optimal curve (representing the best achievable price for a given order size, from a given supply starting point). As noted above, these best-achievable prices constitute the *limiting TBC*.

Note on the Limiting TBC

The optimal curve (or, alternatively, the Limiting TBC), representing the best achievable price for a given trade, assuming infinitesimal trader splitting, has a closed-form expression, already given above:[17]

$$p_0 + \frac{1}{2}\left(\frac{x}{x+C} - \frac{x_0}{x_0+C}\right) + \frac{1}{2}\ln\left(\frac{x+C}{x_0+C}\right)$$

where p_0 is the spot price, x_0 is the current outstanding supply, and x is the *next* outstanding supply value (i.e., the one resulting from the buy or sell of $|x - x_0|$ x-token.)

[16] Optimal in the specific sense that the resultant spot price is lowest.

[17] If the reader is interested, we supply a derivation for this formula, which is a consequence of Riemann sums, in Appendix A.4.

Note that this optimal curve, plotted as the dashed line below, is **not** the ALTBC. It represents an ideal (but fundamentally not achievable) price for a given order size, from the reference point of a single supply position. In practice, every trade moves the ALTBC up or down a given $f_n(x)$ price function, which will yield something very different from the optimal curves shown in the plots below.

Analysis of Incentive to Split

How does the incentive to split work in practice? Two factors matter most here: the number n of partitions, and the starting supply position x from which the partitioned trade is initiated. Obviously, as n grows, the difference between the optimal price and the achieved price decreases (that is the definition of the incentive to split). However, note that this difference also decreases as x grows.

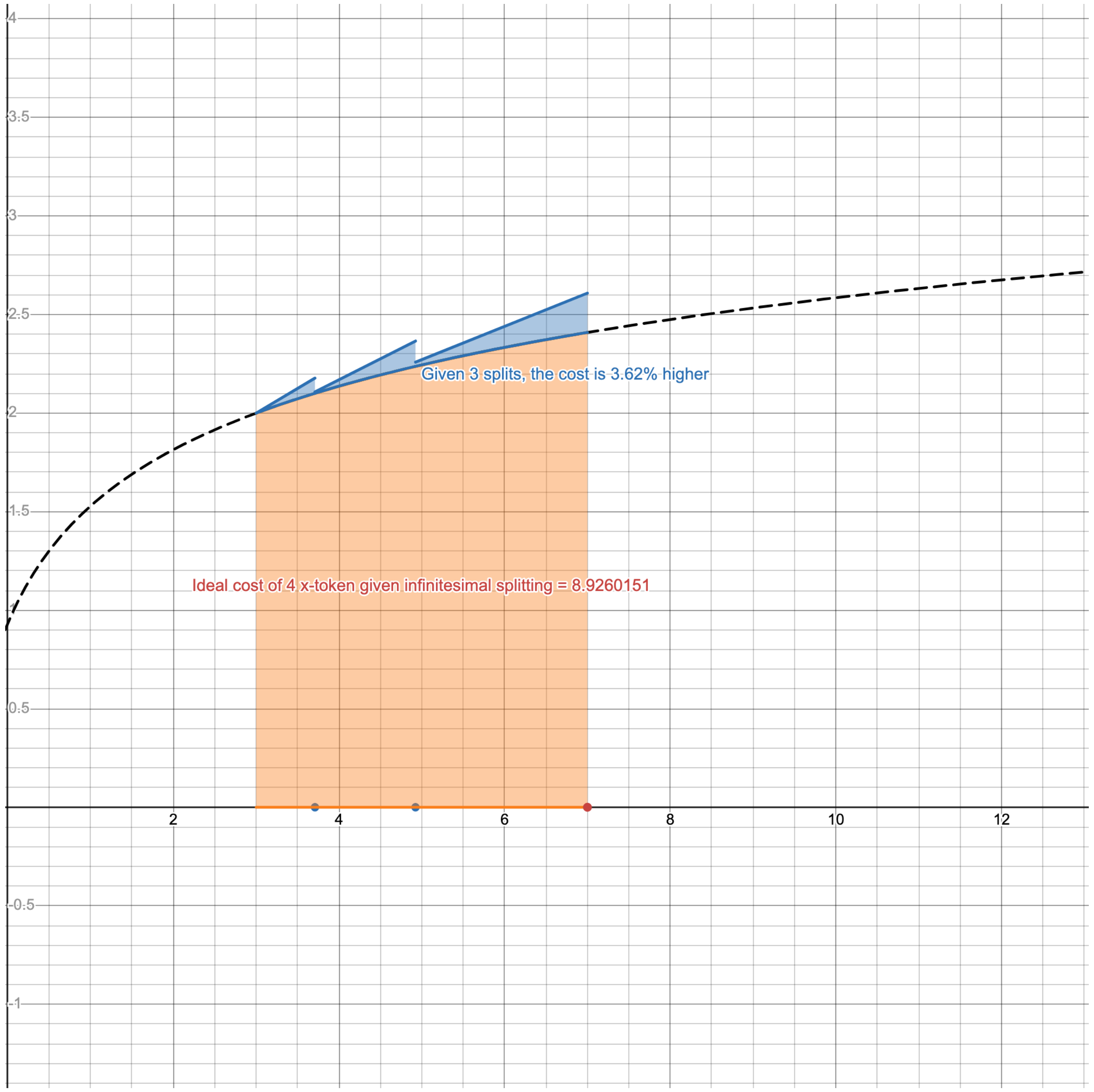


Figure 10: In a low supply setting, 3 partitions lead to a price that is 3.62% higher than optimal.

Figures 10 and 11 show this effect. Let us fix the partition size at 3. In figure 10, we see this partition strategy at work in a relatively low supply situation (initial $x = 3$). These three partitions yield a realized price that is 3.62% higher than the ideal price. Traders might well, in this scenario, feel strongly about how many partitions they are allowed.

In figure 11, on the other hand, we are in a higher supply situation (initial x = 30). Here, the same partition strategy achieves a much closer-to-optimal price outcome. What this means is that, as supply

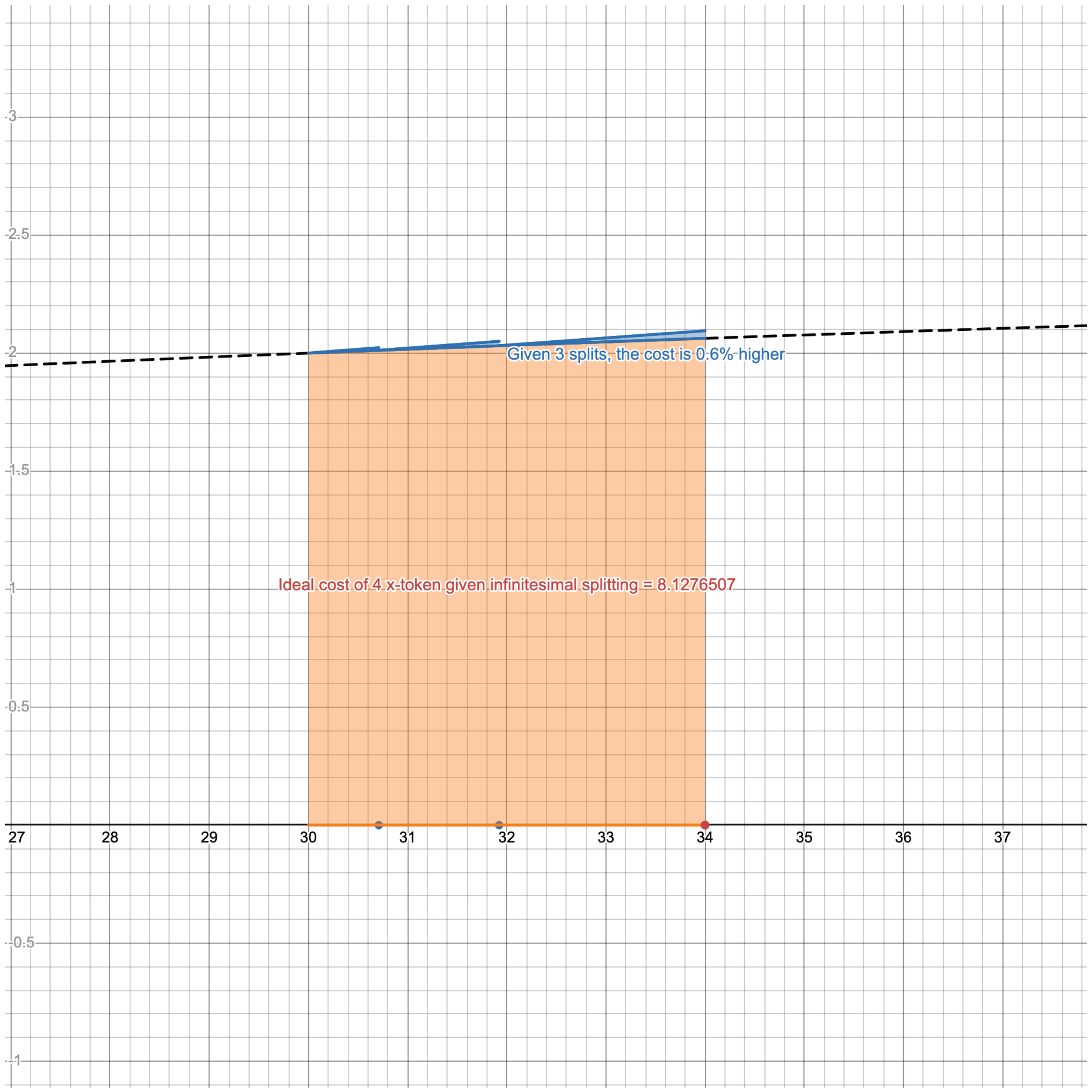


Figure 11: In a higher supply setting, the price after three partitions is already close to the optimal price.

increases, traders care less and less about trying to optimize their partitions, and the incentive will be weaker.

Thus, the incentive to split only really applies in relatively low-supply settings. Even in those settings, however, traders are unlikely to actually deploy complex partition schemes. This is because such schemes take time and deliberation, and because at lower supply, the volatility of the asset is always highest. Given the higher price volatility and greater competition in the low supply setting, the risk of destructive levels of partition optimization is relatively low.

Optimal Order Partitions are Not Uniform

A note on the optimal orders above: while the optimal curve is achievable with uniform infinitesimal splitting, for any given number of partitions the order sizes are non-uniform; hence the uneven size of the partitioned orders in figures 10 and 11. Attackers wishing to obtain optimal order splits face a non-trivial analysis problem, which they would have to execute quickly in the context of the high pressure that often attends assets traded in low-supply circumstances. We note this here because this mathematical obstacle is one of the things that makes implementations of optimal splitting less likely in real life.[18] This is especially true in low-supply settings (the only settings where such schemes really matter), when traders are jockeying for x-token.

Fees and Rules

Note that the above analysis makes no assumptions on ecosystem rules or protocol fees. The incentive to split will be significantly mitigated by gas fees alone. Furthermore, simple rules imposing caps on order frequency per address, or a minimum allowable trade size, could also remove this incentive altogether.

3.6 Notes on $x_{\min}$

The presence of a minimum x-value greater than 0 ensures that prices cannot be costlessly manipulated by exploiters. That is to say, if traders **could** push outstanding supply to zero, there might arise situations where a high volume of round trips to and from $x = 0$ would push the price up and up, without costing the trader anything (other than transaction fees).[19]

$x_{\min}$ imposes a cost on these round trips, so that a malicious actor seeking to inflate prices early on in the ALTBC's lifecycle will have to pay to do so.

How much will this price inflation cost the trader? That depends on the value of $x_{\min}$.

The Constant k

We will think of $x_{\min}$ as a percentage of x_{add}, denoted with k. It is useful, then, to know something about how the ALTBC's behavior will change with different values of k.

k presents a trade-off. On one hand, higher values of k impose higher costs to the malicious trader wishing to push prices up. Figure 12 shows the cost to traders of pushing up price by a factor of 11 (arbitrarily selected) as a function of k size.[20]

On the other hand, higher values of k reduce the market's flexibility. A higher k value means an increase in the minimum achievable price. It also means that the initial slope value at the beginning of the ALTBC's lifespan is lower (meaning prices will move less, earlier on). Finally, it means that, fixing a given trade size, the size of the update to the ALTBC will be smaller. These three effects are shown in figures 13 through 15. Note the three measures of the ALTBC's flexibility in green: initial price, initial slope, and update size (as a percent change in slope from one line to the other). Initial price goes up, while initial slope and update size go down with increases to k. Note also that the realizable trading region is shifted to the right, but its size stays the same (a high k value does **not** mean a decrease in the total number of tokens for sale, a value that is stored as x_{add}).

[18]Sandwich attackers can still make gains with less-than-optimal splits; nevertheless the central point is that systematically splitting orders seems like it might be more trouble than it is worth, especially in the early stages of an asset's lifecycle.

[19]Of course, these trades would also not profit the exploiter - the sole gain would be to sabotage the price mechanism by pushing the price up artificially.

[20]We measure the cost of this price increase as a portion of market capitalization, which is defined here as the lowest price on the starting ALTBC multiplied by x_{add}.

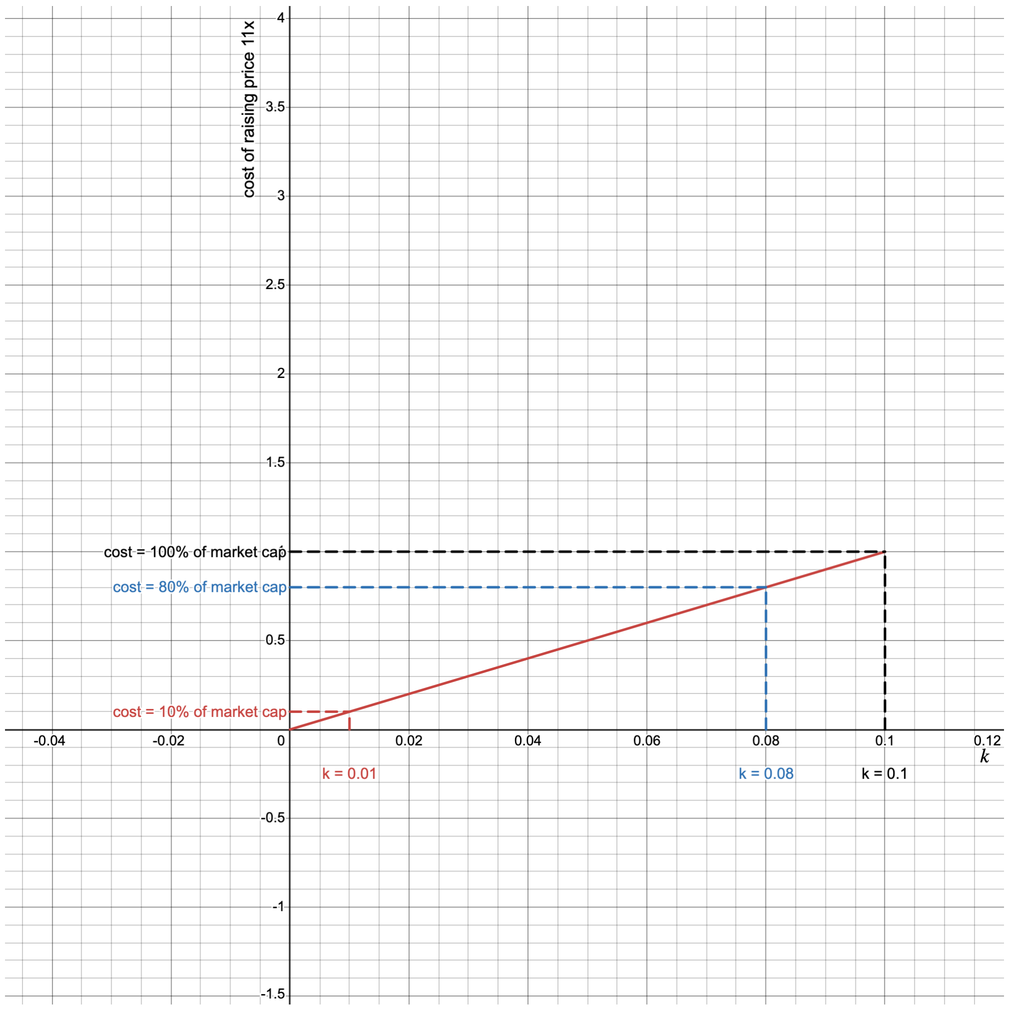


Figure 12: Cost of raising price 11x (measured as a percentage of total market capitalization) as a function of k. To push the price up to 11x its initial value, with $k = .1$, it would cost the trader 100% of market capitalization.

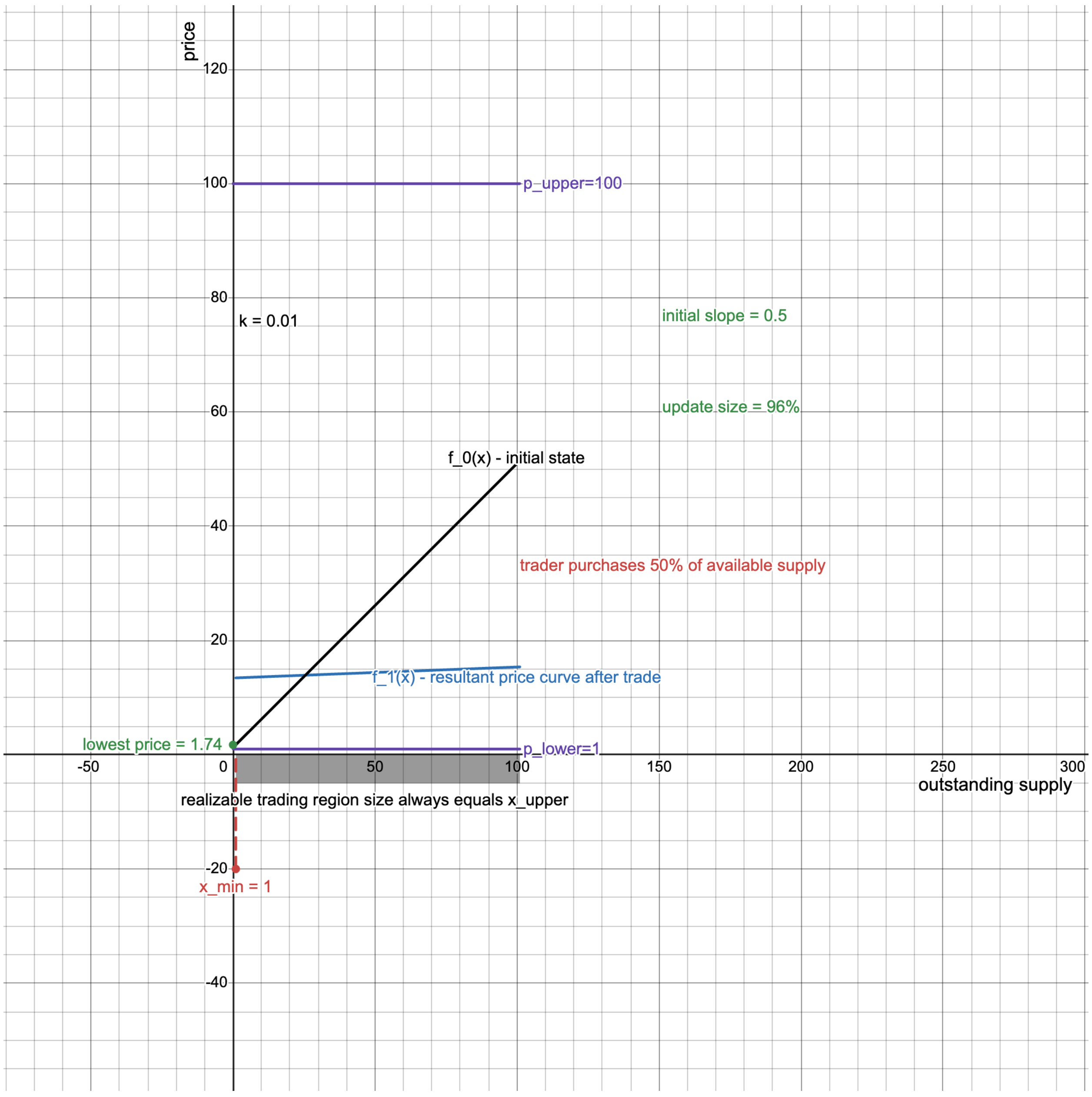


Figure 13: Buying 50% of available supply, $k = .01$. A low k value means a lower initial spot price, a higher initial slope and a more meaningful update to the price curve.

4 Experiments

In this section, we present some experimental results, using real-world trade data to compare the ALTBC with Uniswap pools. In figure 16, we compare the two pools using data from the DOGE-USDT pair. In figure 17, we use TRAX-USDT, and in figure 18, we use ADA-USDT.

4.1 Caveats

There are some caveats we must make before turning to these results:

1. The raw data represent actual buys and sells executed on Uniswap pools (both V2 and V3 are used). These trades took place in the context of actual historical market prices, which differ from the price sequences generated by the ALTBC. Since prices affect demand, however, it might be reasonable to expect different trade sequences to occur in a world where the ALTBC alone was setting prices. In our experiments, we ignore this consideration and hold demand constant regardless of price.

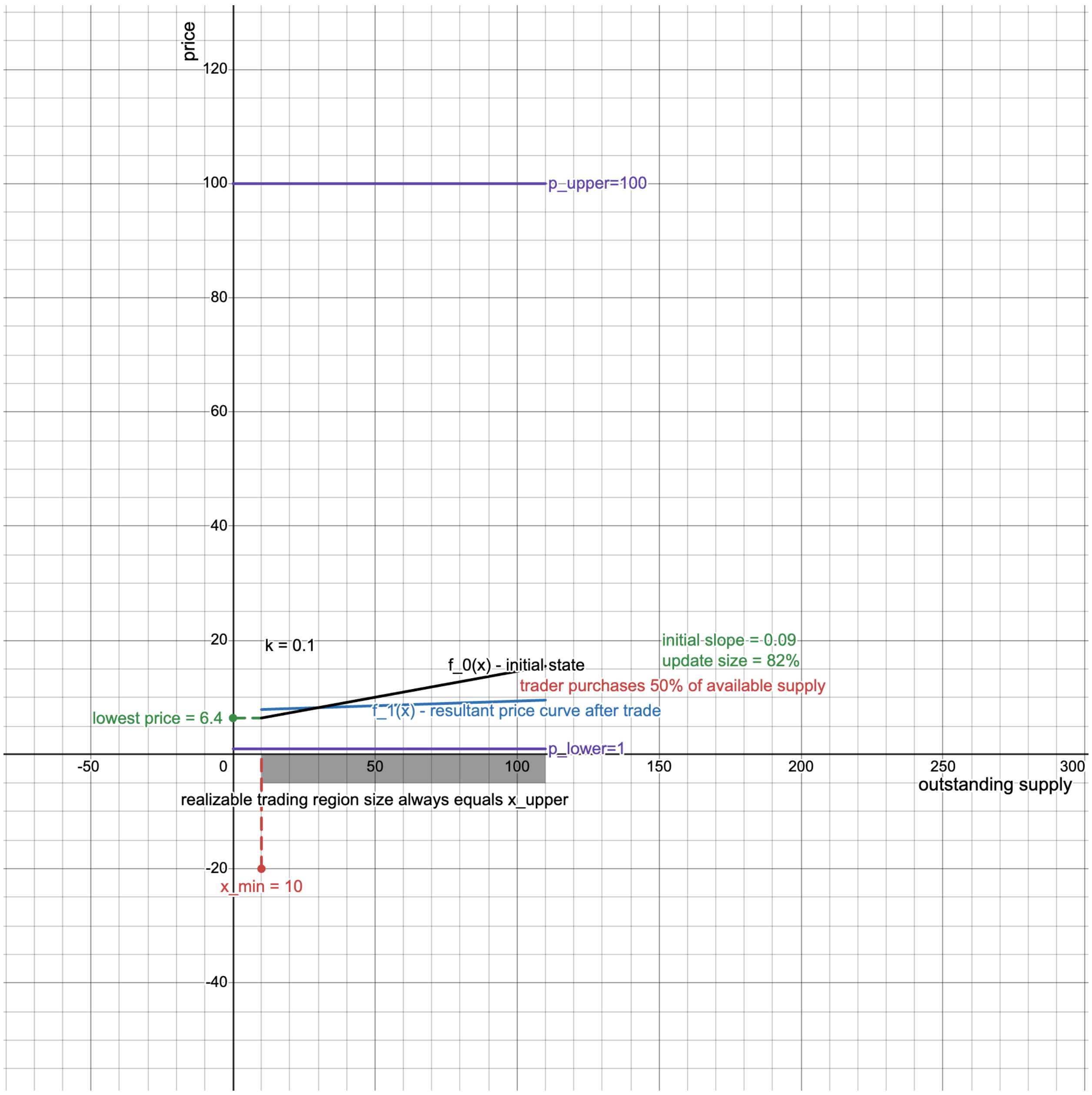


Figure 14: Buying 50% of available supply, $k = .1$. A higher k value means a higher initial spot price, a lower initial slope and a less meaningful update to the price curve after the same trade.

2. The ALTBC can be parameterized in many different ways, which lead to significantly different price behaviors. We have chosen certain parameters that lead to instructive results. The parameters used below are explained in section 2.14 and in the Appendix.

3. In order to accommodate the sequences of trades in the real-world data, we initialize the ALTBC and then run a large initial purchase to prevent the x value from hitting zero and blocking some of the trades in our datasets.

4.2 Discussion

These plots show four things:

1. Whereas Uniswap V2 pools require an initial deposit of collateral, the ALTBC achieves similar price movement without requiring any initial collateral allotment.

2. Whereas Uniswap V3 pools require active management, the ALTBC achieves similar price movements without any active management.

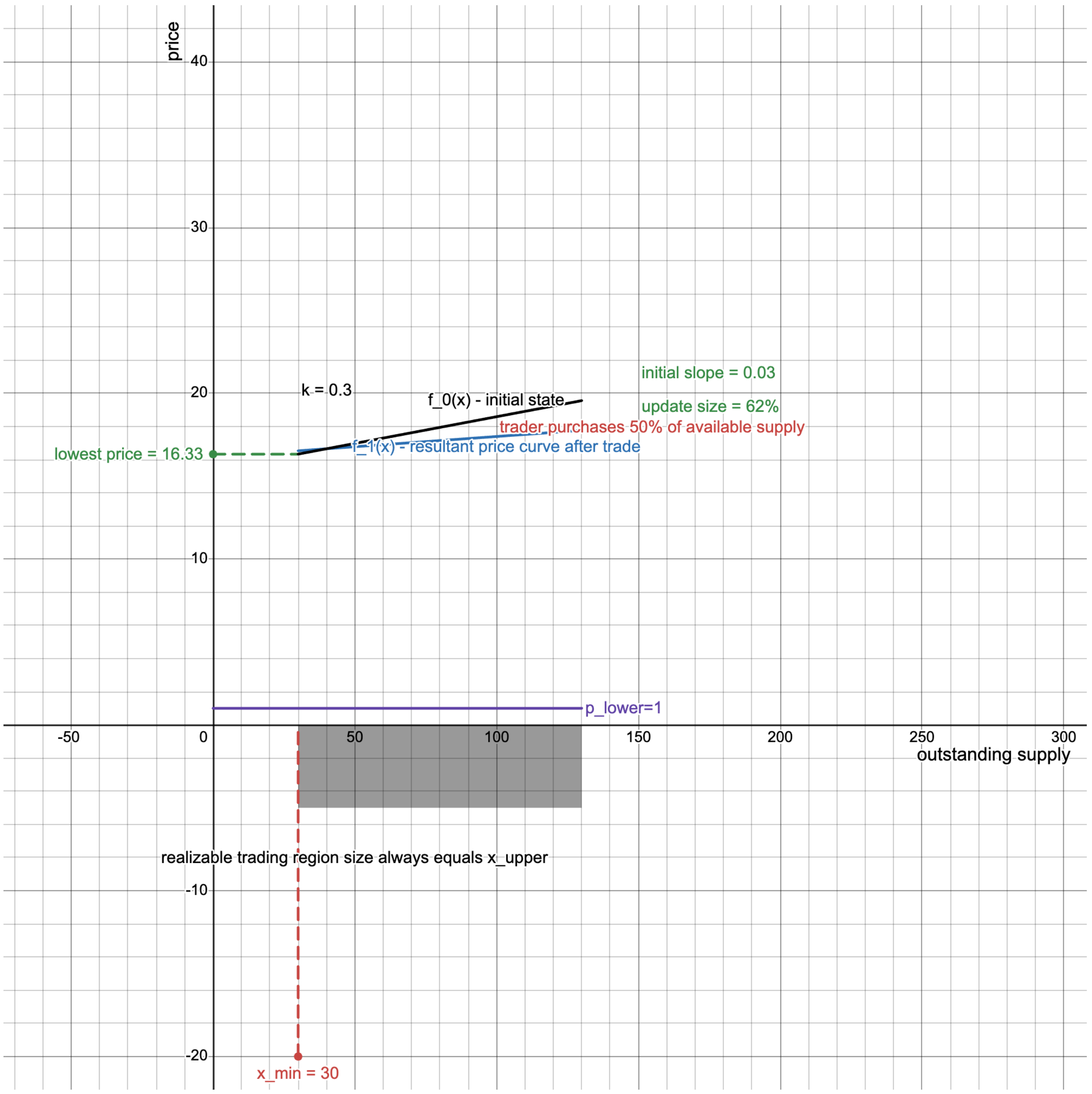


Figure 15: Buying 50% of available supply, $k = .3$. A high k value means a high initial spot price, a low initial slope and a less meaningful update to the price curve after the same trade.

3. The ALTBC has inherent price stability, compared with Uniswap.
4. The `p_mid` value affects this stability, with higher values leading to greater volatility.
5. When volume is high but buys and sells roughly cancel each other in the aggregate, the price and collateral will steadily drift upward.
6. For tokens that trade in the $1 range, the ALTBC is a good combination of bootstrapping utility and price discovery.

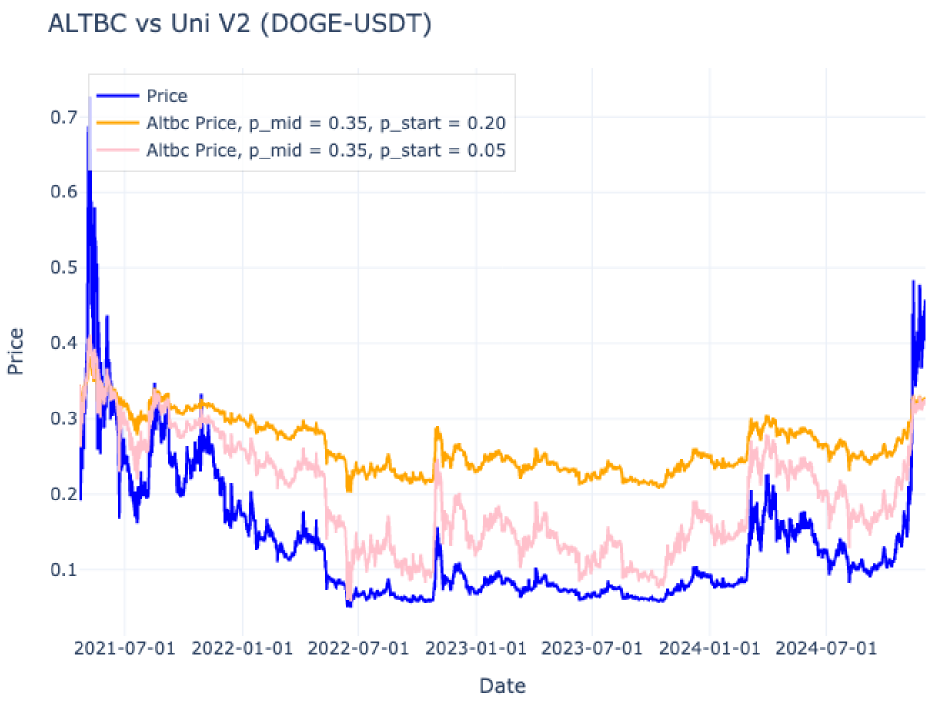


Figure 16: Comparison of ALTBC and UniV2 on the DOGE-USDT trading pair.

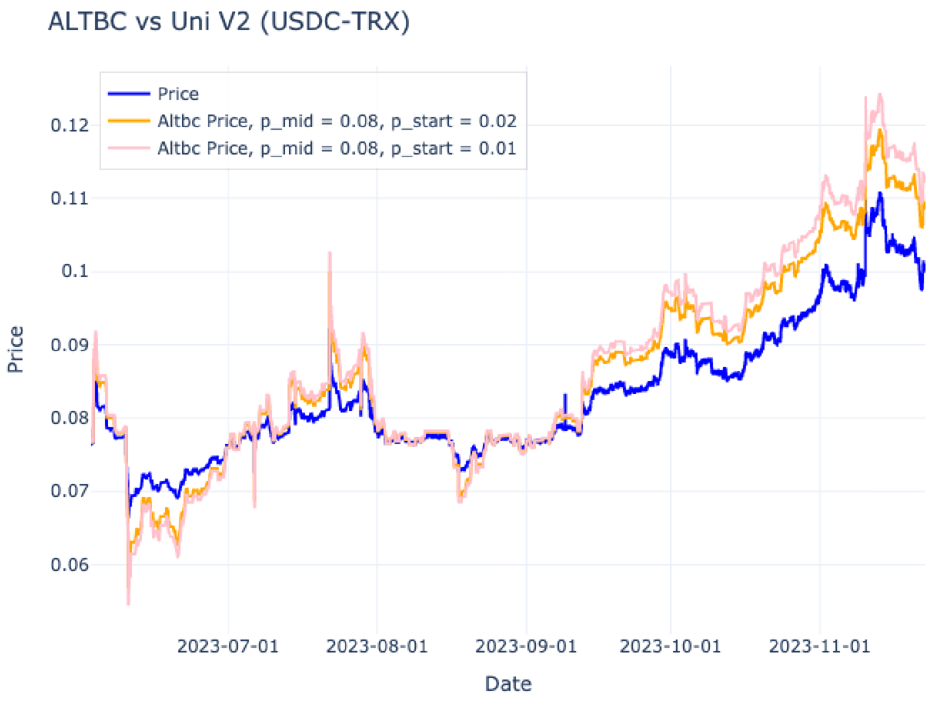


Figure 17: Comparison of ALTBC and UniV2 on the TRX-USDC trading pair.

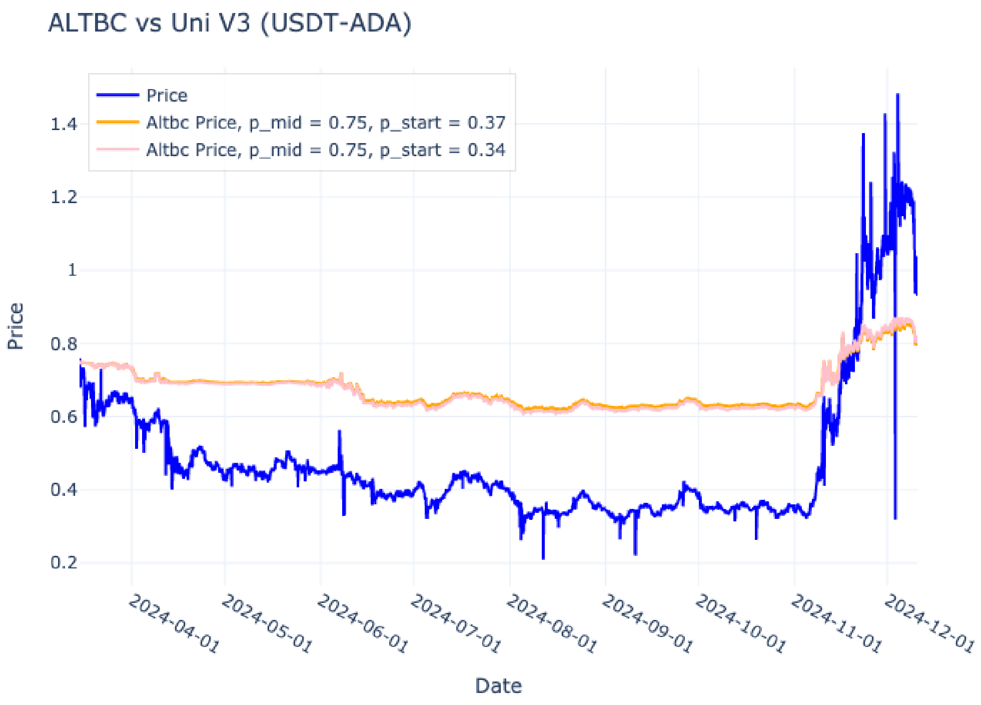


Figure 18: Comparison of ALTBC and UniV3 on the ADA-USDT trading pair.

A Proofs and Derivations

A.1 Formal Description of ALTBC Trading and Price Impact

Initial state. Let x_0 be the amount of tokens sold by the protocol. We consider a linear token bonding curve defined by

$$f(x) = b_0 x + c_0 \ ,$$

where $b_0, c_0 > 0$. Let

$$s(x) = \frac{V}{x + \Gamma_0} \ .$$

We define the *cost function* by

$$F(x) = \frac{1}{2} b_0 x^2 + c_0 x \ .$$

The amount of collateral in the pool is $F(x_0)$.

Trade. Suppose that a trader buys or sells an amount a of tokens. If $a > 0$ we consider that the trader buys a tokens, and if $a < 0$ we consider that the trader sells $-a$ tokens. In any case, the amount of tokens sold by the protocol after the trade is

$$x_1 = x_0 + a \ .$$

The amount of collateral either paid or received by the trader is given by $\Delta = F(x_1) - F(x_0)$. If $\Delta > 0$, then the trader pays that amount and the collateral in the pool increases ($F(x_1) > F(x_0)$), and if this amount is negative, then the trader receives an amount $-\Delta$ of collateral and the amount of collateral in the pool decreases ($F(x_1) < F(x_0)$).

Note that, since F is a strictly increasing function, we obtain that

$$\Delta > 0 \Leftrightarrow F(x_1) > F(x_0) \Leftrightarrow x_1 > x_0 \Leftrightarrow a > 0 \ .$$

State after the trade. After the trade, the amount of tokens sold by the protocol is

$$x_1 = x_0 + a \ .$$

The parameters b and c of the token bonding curve are updated as follows:

$$b_1 = \frac{V}{x_1 + \Gamma_0}$$

and

$$c_1 = \frac{F(x_1) - \frac{1}{2} b_1 x_1^2}{x_1} = \frac{\frac{1}{2} b_0 x_1^2 + c_0 x_1 - \frac{1}{2} b_1 x_1^2}{x_1} = \frac{1}{2}(b_0 - b_1) x_1 + c_0 \ .$$

Thus, the token bonding curve will be updated as

$$f_1(x) = b_1 x + c_1 \ .$$

The amount of collateral in the pool after the trade is

$$F(x_1) = \frac{1}{2} b_0 x_1^2 + c_0 x_1 \ .$$

The cost function will be also updated as

$$F_1(x) = \frac{1}{2} b_1 x^2 + c_1 x \ .$$

If we use the updated cost function to compute the amount of collateral in the pool after the trade we obtain that this amount should be $F_1(x_1)$. With our definition of the updated parameters we get

$$\begin{aligned} F_1(x_1) &= \frac{1}{2} b_1 x_1^2 + c_1 x_1 = \frac{1}{2} b_1 x_1^2 + \left(\frac{1}{2}(b_0 - b_1) x_1 + c_0 \right) x_1 = \frac{1}{2} b_1 x_1^2 + \frac{1}{2} b_0 x_1^2 - \frac{1}{2} b_1 x_1^2 + c_0 x_1 = \\ &= \frac{1}{2} b_0 x_1^2 + c_0 x_1 = F(x_1) \ , \end{aligned}$$

as desired.

Price impact of the trade. The price impact of the trade is given by

$$\frac{|\text{ effective price of the trade} - \text{spot price before the trade }|}{\text{spot price before the trade}} .$$

Thus, we obtain that

$$\begin{aligned}
\text{Price Impact} &= \left|\frac{\frac{F(x_1)-F(x_0)}{a} - (b_0x_0+c_0)}{b_0x_0+c_0}\right| = \left|\frac{F(x_1)-F(x_0)}{a(b_0x_0+c_0)} - 1\right| = \\
&= \left|\frac{\frac{1}{2}b_0x_1^2 + c_0x_1 - (\frac{1}{2}b_0x_0^2 + c_0x_0)}{a(b_0x_0+c_0)} - 1\right| = \left|\frac{\frac{1}{2}b_0(x_1^2-x_0^2) + c_0(x_1-x_0)}{a(b_0x_0+c_0)} - 1\right| = \\
&= \left|\frac{(\frac{1}{2}b_0(x_1+x_0)+c_0)(x_1-x_0)}{a(b_0x_0+c_0)} - 1\right| = \left|\frac{(\frac{1}{2}b_0(x_1+x_0)+c_0)a}{a(b_0x_0+c_0)} - 1\right| = \\
&= \left|\frac{\frac{1}{2}b_0(x_1+x_0)+c_0}{b_0x_0+c_0} - 1\right| = \left|\frac{\frac{1}{2}b_0x_1 + \frac{1}{2}b_0x_0 + c_0 - (b_0x_0+c_0)}{b_0x_0+c_0}\right| = \\
&= \left|\frac{\frac{1}{2}b_0x_1 - \frac{1}{2}b_0x_0}{b_0x_0+c_0}\right| = \left|\frac{\frac{1}{2}b_0(x_1-x_0)}{b_0x_0+c_0}\right| = \left|\frac{\frac{1}{2}b_0a}{b_0x_0+c_0}\right| .
\end{aligned}$$

Therefore, if p_0 is the spot price before the trade we obtain that

$$\text{Price Impact} = \frac{|a|b_0}{2p_0} .$$

A.2 Directional Consistency of Spot Price

Suppose that x_0 is the current outstanding supply and p_0 is the current spot price. Let x_1 and p_1 be the corresponding quantities after the proposed update. Then,

$$\operatorname{sgn}(x_1 - x_0) = \operatorname{sgn}(p_1 - p_0).$$

Proof:

$$\begin{aligned}
p_1 &= p_0 - \frac{x_0}{x_0+C} + \frac{1}{2}\frac{x_1}{x_0+C} + \frac{1}{2}\frac{x_1}{x_1+C} \\
&= p_0 + \frac{1}{2}\frac{x_1-x_0}{x_0+C} + \frac{1}{2}\left(\frac{x_1}{x_1+C} - \frac{x_0}{x_0+C}\right) \\
&= p_0 + \frac{1}{2}\frac{x_1-x_0}{x_0+C} + \frac{1}{2}(h(x_1) - h(x_0)),
\end{aligned}$$

where

$$h(x) = \frac{x}{x+C}$$

is monotonically increasing. As such, if $x_1 > x_0$ (respectively $x_1 < x_0$) in the right hand side of the expression

$$p_1 - p_0 = \frac{1}{2}\frac{x_1-x_0}{x_0+C} + \frac{1}{2}(h(x_1) - h(x_0)),$$

it follows that the RHS will be positive (respectively negative) and so $p_1 > p_0$ (respectively $p_1 < p_0$).

A.3 Monotonicity of Spot Price in time for fixed x

Suppose that x_0 is the current outstanding supply and p_0 is the current spot price. The next time that x_0 is visited (if at all), the spot price p at this time will be greater than p_0.

Proof: Let us first establish some useful notation. Let

$$x_0, x_1, \ldots, x_n = x_0$$

denote the evolution of the outstanding supply as it departs and returns to x_0 for the first time after n steps. Let

$$p_0, p_1, \ldots, p_n = p$$

be the corresponding spot prices. We will need two lemmas.

Lemma 1: If $n = 2$, monotonicity of spot price in time holds.

Proof:

$$\begin{aligned}
p_2 &= p_0 - \frac{x_0}{x_0 + C} + \frac{1}{2}\frac{x_1}{x_0 + C} + \frac{1}{2}\frac{x_1}{x_1 + C} - \frac{x_1}{x_1 + C} + \frac{1}{2}\frac{x_2}{x_1 + C} + \frac{1}{2}\frac{x_2}{x_2 + C} \\
&= p_0 + \frac{1}{2}\frac{x_1 - x_0}{x_0 + C} + \frac{1}{2}\frac{x_2 - x_1}{x_1 + C} + \frac{1}{2}\left(\frac{x_2}{x_2 + C} - \frac{x_0}{x_0 + C}\right) \\
&= p_0 + \frac{1}{2}\frac{x_1 - x_0}{x_0 + C} + \frac{1}{2}\frac{x_0 - x_1}{x_1 + C} \\
&= p_0 + \frac{1}{2}\frac{x_1^2 - 2x_0x_1 + x_0^2}{(x_0 + C)(x_1 + C)} = p_0 + \frac{1}{2}\frac{(x_1 - x_0)^2}{(x_0 + C)(x_1 + C)} > p_0.
\end{aligned}$$

Lemma 2: For a given i, if $x_i < x_{i+1} < x_{i+2}$ (or respectively $x_i > x_{i+1} > x_{i+2}$), omitting the visit to x_{i+1} leaves the AMM with a strictly greater spot price when arriving at x_{i+2}.

Proof: For a given i, assume that $x_i < x_{i+1} < x_{i+2}$ (or respectively $x_i > x_{i+1} > x_{i+2}$). From the proof for lemma 1, we see that

$$\begin{aligned}
p_{i+2} &= p_i + \frac{1}{2}\frac{x_{i+1} - x_i}{x_i + C} + \frac{1}{2}\frac{x_{i+2} - x_{i+1}}{x_{i+1} + C} + \frac{1}{2}\left(\frac{x_{i+2}}{x_{i+2} + C} - \frac{x_i}{x_i + C}\right) \\
&< p_i + \frac{1}{2}\frac{x_{i+1} - x_i}{x_i + C} + \frac{1}{2}\frac{x_{i+2} - x_{i+1}}{x_i + C} + \frac{1}{2}\left(\frac{x_{i+2}}{x_{i+2} + C} - \frac{x_i}{x_i + C}\right) \\
&= p_i + \frac{1}{2}\frac{x_{i+2} - x_i}{x_i + C} + \frac{1}{2}\left(\frac{x_{i+2}}{x_{i+2} + C} - \frac{x_i}{x_i + C}\right),
\end{aligned}$$

where the final expression is the spot price in the AMM had x_{i+1} been omitted from the path. Note that the inequality holds for both cases.

Completion of the proof of the main result: Here it might be useful to include an example graph of the x_i sequence with the abscissa being the index. Consider a global extremum of this graph that occurs at (i, x_i). From lemma 2, we can assume WLOG (by adding an x value as necessary) that $x_{i-1} = x_{i+1}$. In other words, this will only make the trade path worse for the AMM (with a lower final spot price). Then, we can remove x_i and x_{i+1} from the trade path by using Lemma 1. Again, this will only leave the AMM with a lower final spot price. Repeating this process will leave the path only x_0 at the end. As such, we have shown that the final spot price p associated with a given trade path is strictly greater than p_0.

General Lemma: From the above proofs, we note that the spot price associated with an arbitrary sequence $x_0, x_1, \ldots, x_n$ is given by

$$p_n = p_0 + \frac{1}{2}\sum_{1}^{n}\frac{x_i - x_{i-1}}{x_{i-1} + C} + \frac{1}{2}\left(\frac{x_n}{x_n + C} - \frac{x_0}{x_0 + C}\right)$$

A.4 Splitting a trade into N steps

Let x_0 be the initial amount of tokens. We consider an updating linear TBC defined by

$$f(z) = b(x)z + c(x) \ ,$$

where $b(x)$ and $c(x)$ are functions that depend on the amount of tokens x before the trade is performed.

Suppose that we want to buy an amount A of tokens in N steps, where the amount of tokens bought at each step depends on a chosen function g as follows. Let $g\colon [0,1] \to \mathbb{R}$ be a continuous and monotonically increasing function such that $g(0) = x_0$ and $g(1) = x_0 + A$. For each $j \in \{1, 2, \ldots, N\}$ we define

$$x_j = g(\tfrac{j}{N}) \ .$$

Note that $x_N = x_0 + A$. For each $j \in \{1, 2, \ldots, N\}$, the amount of token to be bought at the step j is $x_j - x_{j-1}$. Thus, for all $j \in \{1, 2, \ldots, N\}$, the amount of tokens sold by the protocol after step j is x_j.

For each $n \in \{0, 1, \ldots, N\}$, let p_n be the spot price of the TBC after step n. Similarly, for each $n \in \{1, 2, \ldots, N\}$, let b_n and c_n be the values of the parameters b and c of the TBC after step n. Note that $b_n = b(x_n)$ for all $n \in \{1, 2, \ldots, N\}$.

Applying the formulas given in the first section, we obtain that, for all $n \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}
p_n &= b_n x_n + c_n = b_n x_n + \tfrac{1}{2}(b_{n-1} - b_n)x_n + c_{n-1} = \\
&= b_n x_n + \tfrac{1}{2}(b_{n-1} - b_n)x_n + p_{n-1} - b_{n-1}x_{n-1} = \\
&= p_{n-1} + b_n x_n + \tfrac{1}{2}b_{n-1}x_n - \tfrac{1}{2}b_n x_n - b_{n-1}x_{n-1} = \\
&= p_{n-1} + \tfrac{1}{2}b_{n-1}x_n + \tfrac{1}{2}b_n x_n - b_{n-1}x_{n-1} = \\
&= p_{n-1} + \tfrac{1}{2}b_{n-1}x_n + \tfrac{1}{2}(b_n x_n - b_{n-1}x_{n-1}) - \tfrac{1}{2}b_{n-1}x_{n-1} = \\
&= p_{n-1} + \tfrac{1}{2}b_{n-1}(x_n - x_{n-1}) + \tfrac{1}{2}(b_n x_n - b_{n-1}x_{n-1}).
\end{aligned}$$

With an inductive argument it is not difficult to prove that

$$p_N = p_0 + \sum_{j=1}^{N} \left(\tfrac{1}{2}b_{j-1}(x_j - x_{j-1}) + \tfrac{1}{2}(b_j x_j - b_{j-1}x_{j-1})\right) \ .$$

Thus,

$$\begin{aligned}
p_N &= p_0 + \frac{1}{2}\sum_{j=1}^{N} b_{j-1}(x_j - x_{j-1}) + \frac{1}{2}\sum_{j=1}^{N}(b_j x_j - b_{j-1}x_{j-1}) = \\
&= p_0 + \frac{1}{2}\sum_{j=1}^{N} b(x_{j-1})(x_j - x_{j-1}) + \frac{1}{2}(b_N x_N - b_0 x_0) = \\
&= p_0 + \tfrac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}\sum_{j=1}^{N} b(g(\tfrac{j-1}{N}))(g(\tfrac{j}{N}) - g(\tfrac{j-1}{N})).
\end{aligned}$$

Now, the sum

$$\sum_{j=1}^{N} b(g(\tfrac{j-1}{N}))(g(\tfrac{j}{N}) - g(\tfrac{j-1}{N}))$$

is a Riemann-Stieltjes sum of the integral

$$\int_0^1 b(g(x))\,\mathrm{d}g(x) \ .$$

If g is continuously differentiable then

$$\int_0^1 b(g(x))\,\mathrm{d}g(x) = \int_0^1 b(g(x))g'(x)\,\mathrm{d}x = B(g(1)) - B(g(0)) = B(x_N) - B(x_0) \ ,$$

where the function B is a primitive of the function b.

Therefore,

$$\begin{aligned}
\lim_{N\to+\infty} p_N &= \lim_{N\to+\infty}\left(p_0 + \frac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}\sum_{j=1}^{N} b(g(\tfrac{j-1}{N}))(g(\tfrac{j}{N}) - g(\tfrac{j-1}{N}))\right) = \\
&= p_0 + \frac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}\int_0^1 b(g(x))\,\mathrm{d}g(x) = \\
&= p_0 + \frac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}(B(x_N) - B(x_0)).
\end{aligned}$$

It is interesting to observe that the value of this limit is independent of the chosen function g.

Now, if the function b is monotonically decreasing, it follows that

$$\sum_{j=1}^{N} b(g(\tfrac{j-1}{N}))(g(\tfrac{j}{N}) - g(\tfrac{j-1}{N})) \geq \int_0^1 b(g(x))\, \mathrm{d}g(x) \ .$$

Thus,

$$p_N \geq p_0 + \frac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}\int_0^1 b(g(x))\, \mathrm{d}g(x) \ .$$

Therefore,

$$\inf_{n\in\mathbb{N}} p_N = p_0 + \frac{1}{2}(b(x_N)x_N - b(x_0)x_0) + \frac{1}{2}\int_0^1 b(g(x))\, \mathrm{d}g(x) \ .$$

Now, we want to compute the amount of collateral the trader has to pay in order to buy the amount A of tokens. For each $j \in \{1, 2, \ldots, N\}$, let $F_{j-1}(x)$ be the cost function before step j of the trade. Then, for all $j \in \{1, 2, \ldots, N\}$,

$$F_{j-1}(x) = \frac{1}{2}b_{j-1}x^2 + c_{j-1}x \ ,$$

and the amount of collateral that the trader has to pay in step j is $F_{j-1}(x_j) - F_{j-1}(x_{j-1})$.

Let F_N be defined by

$$F_N(x) = \frac{1}{2}b_N x^2 + c_N x \ .$$

Note that the total amount Y_N of collateral that the trader has to pay is given by

$$\begin{aligned} Y_N &= \sum_{j=1}^{N}(F_{j-1}(x_j) - F_{j-1}(x_{j-1})) = \sum_{j=1}^{N} F_{j-1}(x_j) - \sum_{j=1}^{N} F_{j-1}(x_{j-1}) = \\ &= \sum_{j=1}^{N} F_j(x_j) - \sum_{j=1}^{N} F_{j-1}(x_{j-1}) = F_N(x_N) - F_0(x_0) = \frac{1}{2}b_N x_N^2 + c_N x_N - \frac{1}{2}b_0 x_0^2 - c_0 x_0 = \\ &= (p_N - b_N x_N)x_N + \frac{1}{2}b_N x_N^2 - \frac{1}{2}b_0 x_0^2 - c_0 x_0 = p_N x_N - \frac{1}{2}b_N x_N^2 - \frac{1}{2}b_0 x_0^2 - c_0 x_0 \ . \end{aligned}$$

Since x_0, x_N, b_0, c_0 and b_N are fixed and $x_N > 0$, in order to minimize Y_N it suffices to minimize p_N, which was done previously.

A.5 MEV Formula

The measure of MEV used here is given by the idea of a sandwich attack. If a non-malicious trader submits a transaction of quantity Q then an attacker could "sandwich" either side of this transaction with a buy/sell transaction pair each of size S. If the current outstanding supply of token is x_n at the time the attack is initiated then the profit the attacker attains is given by the following equation for $MEV(S,Q)$:

$$MEV(S,Q) = \frac{SV}{2} \cdot \left(\frac{x_n + Q}{C + x_n + Q + S} + \frac{Q}{C + x_n + S} - \frac{x_n}{C + x_n} \right)$$

There are a few important takeaways to consider here:

- Given a fixed non-malicious transaction quantity Q, the size of $MEV(S,Q)$ for the ALTBC is bounded. This is in sharp contrast to other markets (like Uniswap V2) where $MEV(S,Q)$ goes to infinity as S goes to infinity
- For each Q there is a "critical" value of S above which the MEV will be negative. One show that this value S_{crit} will satisfy the following equation:

$$\frac{x_n + Q}{(C + x_n + Q) + S_{crit}} + \frac{Q}{(C + x_n) + S_{crit}} = \frac{x_n}{C + x_n}$$

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